Lagrange multipliers

Suppose you want to find the local extremum of a function $f(x_1, x_2, x_3, \cdots, x_n)$. Then, we have to find the solution to the following equations:

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0 \tag{1}$$

In other words,

$$\nabla f = 0 \tag{2}$$

Now, suppose the coordinates $(x_1, x_2, \cdots, x_n)$ satisfy i constraints gs as follows:

$$g_1(x_1, x_2, \cdots, x_n) = g_2(x_1, x_2, \cdots, x_n) = \cdots = g_i(x_1, x_2, \cdots, x_n) = 0 \tag{3}$$

Then, let me ask you another question. What is the local extremum of f that satisfy these i constraints?

The trick is to introduce $\lambda_1, \lambda_2, \cdots, \lambda_i$ called "Lagrange multipliers," and find the local extremum of the following quantity F instead of the original f:

$$F = f + \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_i g_i \tag{4}$$

Let's confirm that this trick indeed works. We have:

$$\nabla F = 0, \qquad \frac{\partial F}{\partial \lambda_1} = \frac{\partial F}{\partial \lambda_2} = \cdots = \frac{\partial F}{\partial \lambda_i} = 0 \tag{5}$$

The second condition implies (3). Therefore, the coordinates satisfy the constraints. Moreover, plugging (3) to (4), we see $F = f$, which means that extremizing F is equivalent to extremizing f with the constraints g. Now, we are left with the first equation of (5). Plugging (4) into it, we have:

$$\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots + \lambda_i \nabla g_i = 0 \tag{6}$$

To find the extremum of f we have to solve this equation along with the constraints $g = 0$. Here, λs are extra unknowns that should be found upon solving the above equation.

Problem 1. What is the maximum of $x + 4y$ on the ellipse $x^2 + 9y^2 = 1$? (Hint [1])

Summary

- If you want to find the local extremum of f, with the condition $g_1 = g_2 = \cdots = g_i = 0$, you can do so by extremizing

  $$f + \lambda_1 g_1 + \lambda_2 g_2 \cdots + \lambda_i g_i$$

  where λs here are called Lagrange multipliers.

[1] $f = x + 4y, \quad g = x^2 + 9y^2 - 1$