The Lagrangian formulation of classical mechanics

In the late 17^{th} century, Newton invented Newtonian mechanics, which describes nature very well. About 100 years later, Lagrange invented the Lagrangian formulation of classical mechanics. Lagrange's formulation is totally equivalent to Newtonian mechanics, but is formalized very differently. If you want to solve a difficult mechanics problem, you may do so using either Newtonian mechanics or Lagrangian mechanics. Even though the actual calculations may be very different, the solutions are guaranteed to be the same. In many cases, the Lagrangian method is simpler than the Newtonian one, so there may be a computational advantage in learning the Lagrangian formulation of classical mechanics. More importantly, however, the Lagrangian formulation is the natural framework for quantum field theory and general relativity. These cannot be understood without knowledge of the Lagrangian formulation. In light of this importance, I would like to introduce the Lagrangian formulation of classical mechanics in this article.

In our earlier article "Fermat's principle and the consistency of physics," we explained Fermat's principle: light always travels the path that takes the shortest time. Can we apply Fermat's principle to other objects, such as a stone or a ball? Apparently not. If you throw a ball, the ball doesn't follow the path that takes the shortest time; it eventually falls down rather than traveling along a straight line. However, there is still a way to generalize Fermat's principle so that we can apply it to a ball or a stone.

Before doing so, let me first introduce the terms "extremum" and "extremizing," which are crucial to understanding this article. The first is a collective term for "maximum" and "minimum." The latter is a collective term for "maximizing" and "minimizing." In other words, a value is an extremum if it is a maximum or a minimum, and is extremized if it is maximized or minimized. An important property of the extrema of a function is that the first derivative takes the value zero. Using this terminology, we can re-state Fermat's principle by saying that light travels along the path that extremizes the time it takes.

Now, let's move on to our case. 19^{th} Irish physicist, Hamilton noticed that a particle, such as a ball as we mentioned, moving along the path given by the Newtonian equations of motion extremizes, instead of the time it takes, the following quantity, called "action."

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$$S = \int_a^b L dt \tag{1}$$

where t is time, and L is the Lagrangian which is given by

$$L = T - V \tag{2}$$

where T is kinetic energy, and V potential energy. Notice that Lagrangian L is different from the definition of energy which is the sum of kinetic energy and potential energy. The Lagrangian is the difference between kinetic energy and potential energy, not their sum.

Anyhow, there is an important precondition which a particle must satisfy when you find the path that extremizes (1). If we denote coordinates as a function of time by $q(t)$s, then $q_1(a),\ q_2(a),\ \ldots q_n(a),\ q_1(b),\ q_2(b), \ldots q_n(b)$ must be fixed. In other words, the initial time a, the initial position $q(a)$s, the final time b and the final position $q(b)$s are fixed. See Fig. 1 for examples of paths that satisfy these boundary conditions.

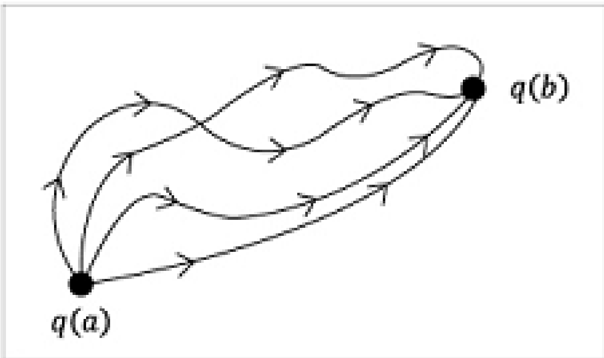

Fig.1

So, among all possible paths that we can draw in Fig. 1 (i.e., the ones that satisfy the boundary condition), the acutal path (i.e., the one that satisfies the Newtonian equations of motion) is the one that extremizes the action (1). That's what Hamilton proved.

Now, let me make two points about (2). If L were just a constant, then (1) would have been just $S = L \int dt$, as we can always pull out a constant factor from an integration. In such a case extremizing S would be just extremizing $L \int dt$ or equivalently, $\int dt$, which is the time that takes. This is the case of light, namely, Fermat's principle.

Second point. Notice also that L, as denoted in (2), is a function of $q(t)$s and $\dot{q}(t)$s since T is a function of $\dot{q}(t)$ (kinetic energy is a function of velocity), and V is a function of $q(t)$ (potential energy depends on position). In other words, we can write

$$L = L(q_1(t), q_2(t), \cdots, q_n(t), \dot{q}_1(t), \dot{q}_2(t), \cdots, \dot{q}_n(t)) \tag{3}$$

Given this, let's find the path that extremizes the action (1) and check whether is agrees with Newtonian equation of motion.

However, finding such a path may seem daunting. In the case of finding simple extrema of ordinary functions, the only thing you need to do is take the derivative with respect to all parameters that describe the function, set them equal to zero, and solve. By contrast, here we have infinitely many such parameters. Namely, S depends on the $q(t)$s with t taking on every value between a and b. Since we have infinitely many such ts, and therefore infinitely many $q(t)$s, we indeed have infinitely many parameters. So, do we need to solve infinitely many equations to find this path?

The answer is no. In the 1750s Euler and Lagrange came up with a novel idea to solve such problems which is called "calculus of variations." Their equation is called the "Euler-Lagrange equation," which we will now derive.

First, for simplicity of notation let action be expressed in the following way:

$$S = \int_a^b L\left(Q_i\left(t\right), \dot{Q}_i\left(t\right)\right) dt \tag{4}$$

where i runs from 1 to n. Now, let $f(t)$ be an arbitrary function that satisfies the boundary condition $f\left(a\right) = f\left(b\right) = 0$ and parametrize $Q_i^{\epsilon}\left(t\right)$ about a certain path $q_i(t)$ with a parameter ϵ as follows:

$$Q_i^{\epsilon}\left(t\right) = q_i\left(t\right) + \epsilon f(t) \tag{5}$$

The above mentioned boundary condition guarantees that $Q_i^{\epsilon}\left(a\right)$ and $Q_i^{\epsilon}\left(b\right)$ are fixed. See Fig. 2.

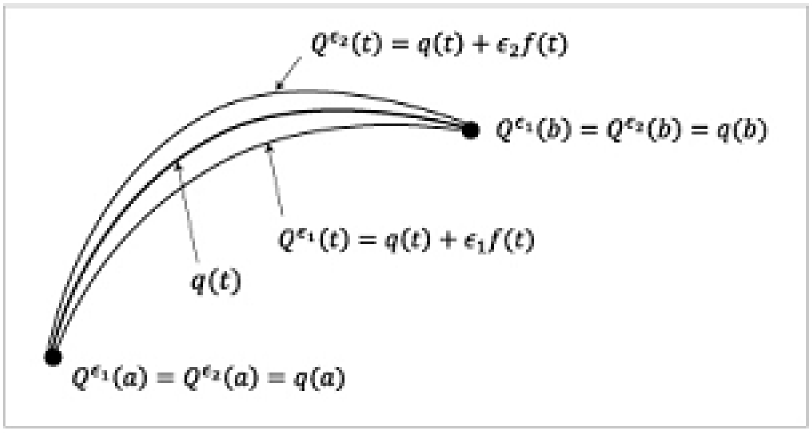

Fig.2

The corresponding action is given by the following formula:

$$S(\epsilon) = \int_a^b L(Q_i^{\epsilon}(t), \dot{Q}_i^{\epsilon}(t)) dt \tag{6}$$

Notice that if $q_i(t)$ extremizes the action, then the following equation must be satisfied regardless of $f(t)$.

$$\frac{dS}{d\epsilon}(\epsilon = 0) = 0 \tag{7}$$

This is because any slight perturbation (i.e. $\epsilon f(t)$ for ϵ very small) around $q_i(t)$ must either increase S if S is a minimum or decrease S if S is a maximum.

Now, let's expand the above equation:

$$\begin{aligned}
\frac{dS}{d\epsilon} = \int_a^b \frac{dL}{d\epsilon} dt &= \int_a^b \left[f(t) \frac{\partial L\left(Q_i^\epsilon(t), \dot{Q}_i^\epsilon(t)\right)}{\partial Q_i^\epsilon(t)} + \dot{f}(t) \frac{\partial L\left(Q_i^\epsilon(t), \dot{Q}_i^\epsilon(t)\right)}{\partial \dot{Q}_i^\epsilon} \right] dt \\
&= \int_a^b \left[f(t) \frac{\partial L}{\partial Q_i^\epsilon} - f(t) \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i^\epsilon}\right) \right] dt + \left[f(t) \frac{\partial L}{\partial \dot{Q}_i^\epsilon} \right]_a^b \qquad (8) \\
&= \int_a^b \left[\frac{\partial L}{\partial Q_i^\epsilon} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i^\epsilon}\right) \right] f(t) dt \qquad (9)
\end{aligned}$$

where we have used integration by parts and the fact that $f(a) = f(b) = 0$. Since for any $f(t)$, the above is satisfied when ϵ is zero, we conclude that the following equation is satisfied for a path that extremizes the action:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{10}$$

This equation is called the "Euler-Lagrange equation," and should be satisfied for all i from 1 to n. In other words, if the system is described by n coordinates, then there are n equations.

Now, Lagrange's observation was that we can retrieve the Newtonian equations of motion by plugging the following Lagrangian into the Euler-Lagrange equation:

$$L = T - V \tag{11}$$

Here, T is kinetic energy, and V potential energy as we explained earlier.

Let's explicitly verify this claim. For kinetic energy and potential energy we have the following:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{12}$$

$$V = V(x, y, z) \tag{13}$$

If we plug this into the Euler-Lagrange equation, we get:

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}, \ \frac{\partial L}{\partial y} = -\frac{\partial V}{\partial y}, \ \frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z} \tag{14}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x} \tag{15}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = \frac{d}{dt}(m\dot{y}) = m\ddot{y} \tag{16}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = \frac{d}{dt}(m\dot{z}) = m\ddot{z} \tag{17}$$

Putting everything together, we finally get

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial V}{\partial x} \tag{18}$$

$$\frac{d}{dt}(m\dot{y}) = -\frac{\partial V}{\partial y} \tag{19}$$

$$\frac{d}{dt}(m\dot{z}) = -\frac{\partial V}{\partial z} \tag{20}$$

which are Newton's equations of motion.

We derived this using Cartesian coordinates, but this agreement with Newton's equations of motion is preserved even if we use other coordinate systems such as polar coordinates, cylindrical coordinates, spherical coordinates and so forth. The reason is following: no matter which coordinate system you use to express the Lagrangian, the path coming from the solutions of the Euler-Lagrange equations for this Lagrangian is guaranteed to extremize the action. Since the actual paths that extremize the action clearly do not depend on a particular coordinate system, we can conclude that the Euler-Lagrange equation for a suitable Lagrangian reproduces Newton's equations of motion.

Finally, I want to add four comments.

First, the notation for the derivation of the Euler-Lagrange equation presented in this article is not the notation one would find in a physics textbook, though it is arguably more conceptually clear. Therefore, I present here the notations commonly used by physicists. The derivation is exactly the same; only the notation is different.

$$0 = \delta S = \int_a^b \delta L dt = \int_a^b \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right] dt \tag{21}$$

$$= \int_a^b \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q dt + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_a^b \tag{22}$$

$$= \int_a^b \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q dt \tag{23}$$

where we have used the following conditions as before: $\delta q\,(t = a) = \delta q\,(t = b) = 0$ Since this equation should be satisfied for arbitrary δq, we retrieve the Euler-Lagrange equation.

Second, a total derivative term in the Lagrangian doesn't affect the equation of motion. In other words, the Euler-Lagrange equations for the Lagrangian L and for the

Lagrangian $L + \frac{dF}{dt}$ are the same. Let's verify this by taking the first steps of derivation of the Euler-Lagrange equation for the second case:

$$0 = \delta S = \int_a^b [\delta L + \delta \dot{F}] dt = \int_a^b \delta L \ dt + [\delta F]_a^b \tag{24}$$

$$= \int_a^b \delta L \ dt + \left[\frac{\partial F}{\partial q} \delta q + \frac{\partial F}{\partial \dot{q}} \delta \dot{q} \right]_a^b \tag{25}$$

$$= \int_a^b \delta L \ dt \tag{26}$$

where in the last step, we used the fact that δq and $\delta \dot{q}$ vanish when $t = a$ or $t = b$. (It goes without saying that $\delta \dot{q}$ vanishes at these points, given that δq does.)

Looking at the above expressions, it is clear that finding the Euler-Lagrange equation for the Lagrangian $L + \frac{dF}{dt}$ is reduced to the problem of finding the Euler-Lagrange equation for the Lagrangian L. Therefore, we can easily conclude that the total derivative term doesn't affect the Euler-Lagrange equation, and the Euler-Lagrange equations for both Lagrangians are the same as claimed.

Third, as most students learn the Lagrangian formulation of classical mechanics long after they first learned the fact that the total energy of a system, such as the sum of kinetic energy and potential energy, is always conserved, they may think that the Lagrangian formulation of classical mechanics is less fundamental than the classical mechanics described by the energy conservation. Of course, as these two formulations are equivalent, we may not be able to say which one is more fundamental. Nevertheless, the modern viewpoint is that Lagrangian formulation is more fundamental; what physicists do is not deriving the form of Lagrangian from the energy, but guessing possible forms of Lagrangian that satisfy stringent criteria, then obtaining the energy that should be conserved (called "Hamiltonian") from the form of Lagrangian. For example, the Lagrangian formulation of general relativity was discovered in 1915 while the Hamiltonian formulation of general relativity was discovered in 1959. In our later articles, we will soon talk about how to obtain the form of Hamiltonian from the form of Lagrangian.

Fourth, earlier in this article, I said, "a ball doesn't follow the path that takes the shortest time." But, this is a lie. A ball does follow the path that takes the shortest time. Nevertheless, if you don't know Einstein's theory of relativity, it's much less confusing to say that "a ball doesn't follow the path that takes the shortest time" than to say that "a ball follows the path that takes the shortest time," even though the former is a lie and the latter isn't. I will explain this point in our later article "Geodesics in the presence of constant gravitational field."

Problem 1. Consider the shortest path that connects the point (x_1, y_1) and (x_2, y_2). This path can be expressed as $y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$. The length element is given by $ds = \sqrt{dx^2 + dy^2}$ from Pythagoras theorem. Then, the quantity we want to extremize is given by

$$\int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} dx \tag{27}$$

By solving the Euler-Lagrange equation, show that the shortest path between these two points satisfies the following equation:

$$\frac{\partial y}{\partial x} = \text{constant} \tag{28}$$

In other words, it's a straight path as the slope seen in an $x - y$ coordinate (Cartesian coordinate) is constant. When we will explain general relativity, we will see that one uses a similar strategy to find the shortest path between two points in a curved space; we will solve the appropriate Euler-Lagrange equation. In the curved space case, $ds = \sqrt{dx^2 + dy^2}$ is modified; if $ds = \sqrt{dx^2 + dy^2}$ is satisfied at all points, the space is necessarily flat.

Summary

- Lagrangian is given by $L = T - V$ where T is kinetic energy and V is potential energy.
- The integration of Lagrangian with respect to time is the action S.
- When a particle moves, its action is always extremized. i.e. $\delta S = 0$.
- The Euler-Lagrange equation is given by

  $$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

  where q_i are coordinates.