Lorentz transformation

1 Introduction

Let's say that an observer in the reference frame S observes that an event A took place on the coordinate (x, y, z, t), and another observer in the reference frame S' moving relative to S in the x-direction with a constant speed v observes that the same event A took place on some coordinate (x', y', z', t'). (Here x, y, z denote the three dimensional coordinates of location, and t the time.) There should be some relation between them. To this end, we will assume that both reference frames are inertial (i.e. an object at rest in an inertial reference frame remains at rest unless other external force is exerted), and when the observer in S observes that an event took place at $(x, y, z, t) = (0, 0, 0, 0)$, the observer in S' will observe that the same event took place at $(x', y', z', t') = (0, 0, 0, 0)$. In other words, the two observers were at the same point (i.e. $(x, y, z) = (x', y', z') = (0, 0, 0)$) when $t = t' = 0$. We will obtain the relation in the non-relativistic case first, then in the relativistic case. The former is called "Galilean transformation" and the latter "Lorentz transformation."

2 Galilean transformation

The Galilean case is easy. As S' moves by the amount vt during t seconds, we have:

$$x' = x - vt \tag{1}$$

And, as the other directions are unaffected, we have:

$$y' = y \tag{2}$$
$$z' = z \tag{3}$$

Moreover, we know that non-relativistically time always flows at the same rate.

$$t' = t \tag{4}$$

Given this, how is the velocity observed by the observer in S' related to the one observed by the observer in S? This is easy. First, notice that we have the following "infinitesimal" Galilean transformation[1]

$$
\begin{aligned}
dx' &= dx - vdt && (5)\\
dy' &= dy && (6)\\
dz' &= dz && (7)\\
dt' &= dt && (8)
\end{aligned}
$$

Therefore, for the velocity, we have:

$$
\begin{aligned}
\frac{dx'}{dt'} &= \frac{dx}{dt} - v && (9)\\
\frac{dy'}{dt'} &= \frac{dy}{dt} && (10)\\
\frac{dz'}{dt'} &= \frac{dz}{dt} && (11)
\end{aligned}
$$

Of course, the above equations are wrong on a relativistic scale, as we know that the speed of light should always be constant. It is easy to see that the constancy of the speed of light is not respected in the above equation. If light moves at the speed of c with respect to the reference frame S, then the observer in the reference frame S' will observe the speed of light to be $c-v$, as we can see from (9). Therefore, we need to find a relativistically correct transformation, called the "Lorentz transformation." This is the subject of the next section.

3 Lorentz transformation

Let's guess how (1) should be modified. Apparently, the observer at the origin of S' (i.e. $(x' = 0, y' = 0, z' = 0, t')$) will be seen as $(x = vt, y = 0, z = 0, t)$ by the observer in the reference frame S, since S' is moving at the speed v in the x-direction. So we guess:

$$x' = \gamma(x - vt) \tag{12}$$

where γ is a certain positive constant that depends on v. It is easy to see that $x = vt$ yields $x' = 0$. Notice that the above equation also satisfies the reasonable requirement that the bigger the x, the bigger the x' if t is the same. This requirement is reasonable for the following reason: If an observer in S sees that John is placed on the right side of Michael, an observer in S' must agree with this as well, as long as they both see the positive x-direction as the rightward direction.

[1] If you don't understand these expressions, just think of d as a calculus way of expressing Δ. For example, $\Delta x' = \Delta x - v\Delta t$, $\Delta y' = \Delta y$ and so on.

On the other hand, the y and z coordinate will be unaffected, since S' is not moving along either of these directions. So, we will simply have $y' = y$ and $z' = z$ as before.

Now, observe that to an observer at S', it is S that moves with speed v. So we have:

$$x = \gamma(x' + vt') \tag{13}$$

Compare this equation with (12). We have $+vt'$ instead of $-vt'$ since S is moving in negative x-direction. Now plug (12) into (13). We get:

$$t' = \gamma t + \left(\frac{1-\gamma^2}{\gamma v}\right) x \tag{14}$$

Given this, use the condition that the speed of light is constant. This implies that $x = ct$ should be mapped to $x' = ct'$. From (12) and (14), we have:

$$\gamma(x - vt) = c\gamma t + \left(\frac{1-\gamma^2}{\gamma v}\right) cx \tag{15}$$

Solving for x:

$$x = ct\left(\frac{1+\frac{v}{c}}{1-(\frac{1}{\gamma^2}-1)\frac{c}{v}}\right) \tag{16}$$

So, the term in the parentheses must be 1. Solving this again, we get:

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \tag{17}$$

Therefore, the Lorentz transformation is given by:

$$x' = \gamma(x - vt) \tag{18}$$
$$y' = y \tag{19}$$
$$z' = z \tag{20}$$
$$t' = \gamma(t - \frac{vx}{c^2}) \tag{21}$$
$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \tag{22}$$

What would be the formula for the inverse Lorentz transformation (i.e. the expression for (x, y, z, t) in terms of (x', y', z', t'))? We can just solve the above equations, taking (x, y, z, t) as the unknown variables and (x', y', z', t') as the known ones. However, there is an easier way that yields the same answer. Remember our earlier discussion that S is the one that is moving from the point of view of S'. So we exchange the variables between S and S', and replace v by $-v$. This yields:

$$x = \gamma(x' + vt') \tag{23}$$
$$y = y' \tag{24}$$
$$z = z' \tag{25}$$
$$t = \gamma(t' + \frac{vx'}{c^2}) \tag{26}$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{27}$$

4 Addition of velocity

Let's say that an object moves with speed $v_x = dx/dt$ in the x-direction with respect to S. What would the observer in S' observe as its velocity? Let's use the Lorentz transformation. We have:

$$dx' = \gamma(dx - vdt) \tag{28}$$
$$dy' = dy \tag{29}$$
$$dz' = dz \tag{30}$$
$$dt' = \gamma(dt - \frac{vdx}{c^2}) \tag{31}$$

which yields:

$$\frac{dx'}{dt'} = \frac{dx - vdt}{dt - \frac{vdx}{c^2}} = \frac{\frac{dx}{dt} - v}{1 - \frac{\frac{dx}{dt}v}{c^2}} \tag{32}$$

Similarly, from the inverse Lorentz transformation, we have

$$\frac{dx}{dt} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{\frac{dx'}{dt'}v}{c^2}} \tag{33}$$

So, this is the relativistic "addition" rule for velocity. Notice that in our everyday lives in which the speed concerned is much smaller than c (we call this case the "non-relativistic limit"), the denominator of the above equation is very close to 1, so we get $dx/dt \approx dx'/dt' + v$. For example, if you run with speed 10m/s from the tail of airplane toward the cockpit of airplane flying with speed 300m/s, an outside observer will see that your speed is given by $309.9999999999897\cdots$ m/s which, indeed, is close to 310m/s. This shows that we can ignore relativistic effects in our everyday lives.

Now, let's also check that the speed of light is constant regardless of the observer. Replacing dx'/dt' by c, we get:

$$\frac{c + v}{1 + \frac{cv}{c^2}} = c \tag{34}$$

Also, even though we will not prove this, it can be shown that the addition of two velocities less than the speed of light must be less than c. For example, $0.9c$"+"$0.9c$ is given as follows:

$$\frac{0.9c + 0.9c}{1 + \frac{0.9c \cdot 0.9c}{c^2}} = \frac{180}{181}c \tag{35}$$

which is less than c.

Problem 1. Let's say that an object moves with speed $v_y = dy/dt$ in the y-direction with respect to S. What would the observer in S' observe as its velocity? Check that the answer becomes $-v\hat{x} + v_y\hat{y}$ in the non-relativistic limit. Check also that if $v_y = c$, the observer in S' observe the object's speed also as c.

5 Time dilation

Let's derive time dilation using the Lorentz transformation. Let's say that a clock is at rest with respect to the frame S, and call its position x_0. This clock records two events that happened at (t_1, x_0) and (t_2, x_0), and measures the duration between the events to be $t_0 = t_2 - t_1$. However, an observer at S' sees that the clock is moving and thinks that the two events happened at the following points:

$$t'_1 = \gamma(t_1 - \frac{vx_0}{c^2}) \tag{36}$$

$$t'_2 = \gamma(t_2 - \frac{vx_0}{c^2}) \tag{37}$$

So, we have:

$$t' \equiv t'_2 - t'_1 = \gamma(t_2 - t_1) = \gamma t_0 \tag{38}$$

So, the time as measured by the moving clock goes more slowly, since $\gamma > 1$

6 Length contraction

Let's derive length contraction using the inverse Lorentz transformation. Let's say that a ruler is at rest with respect to the frame S. Its ends are at x_1 and x_2. Therefore, the intrinsic length is $L_0 = x_2 - x_1$. Now, the observer at the frame S' sees that the first end is at (x'_1, t'_1) and the other end at (x'_2, t'_2). When $t'_1 = t'_2$, the observer measures the length of the ruler to be $L \equiv x'_2 - x'_1$, since you measure the length of a ruler by subtracting the position of one end from that of another recorded *at the same time*. Notice also that in the frame S you didn't need to measure the length of a ruler by measuring the end positions at the same time, since the ruler is not moving anyway. Therefore, we don't need to impose $t_1 = t_2$, while we need to impose $t'_1 = t'_2$. To impose the latter condition, it is more convenient to use the inverse Lorentz transformation than the Lorentz transformation. Thus, using the inverse Lorentz transformation, we have:

$$x_1 = \gamma(x'_1 + vt'_1) \tag{39}$$
$$x_2 = \gamma(x'_2 + vt'_2) \tag{40}$$

Therefore,

$$L_0 = x_2 - x_1 = \gamma(x'_2 - x'_1 + v(t'_2 - t'_1)) = \gamma L \tag{41}$$

Therefore, we conclude

$$L = \frac{L_0}{\gamma} \tag{42}$$

Since γ is bigger than 1, we see that the moving object is contracted.

7 The relativity of simultaneity

We, human beings, have an intuitive understanding of simultaneity. For example, you hear people saying "I was eating ice cream when the terrorists attacked the World Trade Center." In other words, if the event 1 "eating ice cream" occurred at (x_1, y_1, z_1, t_1) and the event 2 "the World Trade Center attack" occurred at (x_2, y_2, z_2, t_2), observed both from the point of view of reference frame S, then the simultaneity of the two events imply $t_1 = t_2$. Now, let me ask you the following question. Did these two events occur at the same time from the point of view of an observer "S'" moving in the x-direction with constant speed v relative to S? In other words, would t'_1 be equal to t'_2? Now, we can use the Lorentz transformation. (21) shows the following:

$$\begin{aligned}
t'_1 &= \gamma(t_1 - \frac{vx_1}{c^2}) && (43)\\
t'_2 &= \gamma(t_2 - \frac{vx_2}{c^2}) && (44)\\
(t'_2 - t'_1) &= \gamma\left((t_2 - t_1) - \frac{v(x_2 - x_1)}{c^2}\right) && (45)\\
&= -\frac{\gamma v(x_2 - x_1)}{c^2} && (46)
\end{aligned}$$

Therefore, we conclude that the observer at S' won't say that the two events occurred at the same time, since $t'_2 - t'_1$ is not zero. (Of course, the two events would have occurred at the same time if $x_2 - x_1 = 0$, but this would imply that you were at the World Trade Center when you were eating ice cream.) Therefore, we conclude that simultaneity is not absolute. It depends on the motion of the observers.

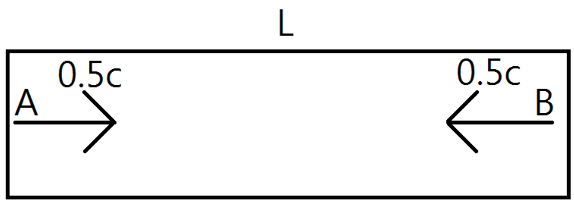
8 Comment

A common mistake in interpreting the Lorentz transformation is that if an observer "observes" an event at (x, t), then the light signal that informs the observer that this event has taken place must have already reached the observer at time t. This is not true. To calculate the time when the observer is informed of this event, to the time t, one should remember to add the time it took for the light signal to reach the observer.

9 Historical Remark

Lorentz, who first derived the Lorentz transformation using formulas that originated from electromagnetic theory, later noted that he nevertheless failed to discover special relativity because he didn't consider the possibility that both times (i.e. t, t') were real; he discriminated between them by calling one "true time" and the other "local time." Moreover, he failed to derive the right equation for the velocity "addition" rule. Einstein derived the Lorentz transformation using the constancy of speed of light in a way similar to the one presented in this article, which no one had done before. Furthermore, Einstein got the velocity "addition" rule right. Lorentz and Einstein were once nominated as candidates for Nobel Prize for their discovery of special relativity, but it was never realized. However, Lorentz won a Nobel Prize long before this nomination, even before Einstein finished the discovery of special relativity, and Einstein later received a Nobel Prize for another work.

Problem 2. See the figure. A big container of length L is at rest. From the reference frame of the container, object A and object B which were initially located at the ends of the container started to move toward one another simultaneously, each with speed $0.5c$. From the point of view of the container, how long did it take for them to meet? How about from the point of view of object A? or object B? From the point of view of object A, which one of the two objects started to move first? And, how much earlier?



Summary

- If we introduce $\beta = v/c$, and use the coordinate ct instead of c, the Lorentz transformation becomes a form easier to remember. We see that the equation is symmetry under the exchange of x and ct.

$$
\begin{aligned}
x' &= \gamma(x - \beta ct) \\
y' &= y \\
z' &= z \\
ct' &= \gamma(ct - \beta x) \\
\gamma &= \frac{1}{\sqrt{1-\beta^2}}
\end{aligned}
$$