Lorentz transformation and Rotation, a comparison

In our article, "rotation in Cartesian coordinate system" we saw that under rotation around the origin the distance from the rotated point to the origin remain same. Recall the following formula:

$$r(x,y) = \sqrt{x^2 + y^2} = r(x', y') = \sqrt{x'^2 + y'^2} \tag{1}$$

This formula can be generalized to a rotation in three-dimensions, which is:

$$r(x,y,z) = \sqrt{x^2 + y^2 + z^2} = r(x', y', z') = \sqrt{x'^2 + y'^2 + z'^2} \tag{2}$$

Actually, it is easy to check that the distance between two points is also invariant under rotation as follows:

$$\Delta r = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2} \tag{3}$$

where $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta x' = x'_2 - x'_1$ and so on.

A similar, but different relation can be found for Lorentz transformation. Given Lorentz transformation, it is an easy exercise to check that following holds:

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y^2) + (\Delta z)^2 - (c\Delta t)^2 = (\Delta x')^2 + (\Delta y'^2) + (\Delta z')^2 - (c\Delta t')^2 \tag{4}$$

In other words, Δs is invariant under Lorentz transformation. Δs is called "proper distance." The above equation holds when the right-hand side is bigger than zero. When it is negative, we have the following:

$$(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y^2) - (\Delta z)^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y'^2) - (\Delta z')^2 \tag{5}$$

In other words, $\Delta\tau$ is invariant under Lorentz transformation. $\Delta\tau$ is called "proper time."

In conclusion, we see that proper distance and proper time are invariant under Lorentz transformation, as much as the distance is invariant under rotation. Therefore, Lorentz transformation can be regarded as "rotation" in 4-dimension. This 4-dimension space is called Minkowski space, named after the mathematician and former teacher of Einstein, who came up with this idea in 1907 two years after Einstein discovered theory of special relativity. We will talk more about this view and Minkowski space in later articles. As an aside, Minkowski recalled Einstein being “a lazy dog” in his student days, expressing his surprise on Einstein’s discovery of relativity.

Final comment. Even though one can derive time dilation formula from Lorentz transformation it is insightful to see how it can be seen from (5). Let’s say that a rocket is moving with constant velocity v in positive x-direction with respect to an observer S. The rocket sees that his own clock elapses $\Delta t'$ while $\Delta x' = 0$ as his own clock is not moving with respect to him. On the other hand, while rocket’s clock elapses $\Delta t'$, S sees that his clock elapses Δt and the position of rocket changes by $\Delta x = v\Delta t$. Therefore, we see:

$$(c\Delta t')^2 - 0^2 = (c\Delta t)^2 - (v\Delta t)^2 \tag{6}$$

which implies

$$\Delta t' = \Delta t\sqrt{1 - \frac{v^2}{c^2}} \tag{7}$$

This is exactly time dilation; The moving clock (i.e. the rocket’s clock) ticks slower rate than the unmoving one (i.e. the observer S) as $\Delta t' < \Delta t$ clearly shows.

Summary

- The distance between two points is invariant under rotation. i.e.

$$\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

- Similarly, the proper time or the proper distance between two points is invariant under Lorentz transformation. i.e.

$$(c\Delta\tau)^2 = (c\Delta t)^2-(\Delta x)^2-(\Delta y^2)-(\Delta z)^2 = (c\Delta t')^2-(\Delta x')^2-(\Delta y'^2)-(\Delta z')^2$$

$$(\Delta s)^2 = (\Delta x)^2+(\Delta y^2)+(\Delta z)^2-(c\Delta t)^2 = (\Delta x')^2+(\Delta y'^2)+(\Delta z')^2-(c\Delta t')^2$$