Newton's first and second laws

In this article, we introduce Newton's first and second laws somewhat mathematically. To understand the physical significance of Newton's first law, readers are encouraged to read my earlier article "Newton's first law."

Newton's first law states that a moving object keeps moving at the same velocity and an object not moving stay at rest unless an external force is exerted. Newton's second law states that the acceleration of an object is proportional to the external force and inversely proportional to the mass of the object. In other words:

$$\vec{a} = \frac{\vec{F}}{m} \tag{1}$$

where $\vec{a}$ is the acceleration, $\vec{F}$ is the force and m is the mass. From this equation, we can easily see that an object's acceleration (i.e. $\vec{a}$) is zero, if the external force $\vec{F}$ is zero. In other words, Newton's first law is a special case of Newton's second law.

Now we will show that the acceleration an observer measures is independent of which inertial reference frame the observer is in.[1] Let's say the inertial reference frame S' moves with the velocity $\vec{v}_0$ relative to the inertial reference frame S. If an observer in S observe an object's velocity to be $\vec{v}$, another observer in S' will observe the object's velocity to be $\vec{v}\,' = \vec{v} - \vec{v}_0$. Then, the observer in S will observe the object's acceleration to be $\vec{a} = d\vec{v}/dt$ while the observer in S' will observe the object's acceleration to be $\vec{a}\,' = d\vec{v}\,'/dt' = d\vec{v}\,'/dt$. (As we are ignoring the relativistic effect, $t' = t$. See our earlier article on Lorentz transformation.) Given this, let's compare these two observed accelerations. We have

$$\vec{a}\,' = \frac{d\vec{v}\,'}{dt} = \frac{d(\vec{v} - \vec{v}_0)}{dt} = \frac{d\vec{v}}{dt} - \frac{d\vec{v}_0}{dt} = \frac{d\vec{v}}{dt} \tag{2}$$

Here, we used the fact that $\vec{v}_0$ is constant, if both S and S' are inertial reference frame. (See "Newton's first law.") Thus, we conclude that the acceleration of an object an observer observes doesn't depend on which inertial reference frame the observer is in, as long as he is in an inertial reference frame. Now, notice that Newton's formula $\vec{F} = m\vec{a}$ doesn't have in it the velocity $\vec{v}$, which depends on the choice of the inertial reference frame; it has only $\vec{a}$ which is independent of which inertial reference frame we are in. Thus, it indeed makes sense that the formula $\vec{F} = m\vec{a}$ doesn't have $\vec{v}$ in it. Otherwise, the law of physics would have depended on which inertial reference frame we are in.

[1] However, this is true only when we ignore the relativistic effect, as you can see from our earlier article "Electromagnetic forces and time dilation in special relativity." There we have seen that acceleration does depend on the inertial reference frame, which leads to time dilation.

Now, let's say that we are in a certain inertial reference frame and consider the case that the external force F is a constant given by $\vec{F_0} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ Then, we can integrate (1) to obtain velocity as follows:

$$\vec{v} = \int \vec{a}dt = \int \frac{\vec{F_0}}{m}dt = \vec{v}_0 + \frac{\vec{F_0}t}{m} \tag{3}$$

where v_0 is the integration constant. By plugging $t = 0$ to the above formula, we can see that $\vec{v}_0$ is the velocity when $t = 0$. Component-wise, the above formula can be re-written as follows:

$$v_x = v_{x0} + \frac{F_x}{m}t \tag{4}$$

$$v_y = v_{y0} + \frac{F_y}{m}t \tag{5}$$

$$v_z = v_{z0} + \frac{F_z}{m}t \tag{6}$$

where $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{v}_0 = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath} + v_{z0}\hat{k}$. By integrating (3) once more, we can get the position of the object as follows:

$$\vec{s} = \int \vec{v}dt = \vec{s}_0 + \vec{v}_0t + \frac{1}{2}\frac{\vec{F_0}}{m}t^2 \tag{7}$$

where $\vec{s}_0$ is the integration constant. By plugging $t = 0$ to the above formula, we can see that $\vec{s}_0$ is the position when $t = 0$. Component-wise, the above formula can be re-written as follows:

$$x = x_0 + v_{x0}t + \frac{1}{2}\frac{F_x}{m}t^2 \tag{8}$$

$$y = y_0 + v_{y0}t + \frac{1}{2}\frac{F_y}{m}t^2 \tag{9}$$

$$z = z_0 + v_{z0}t + \frac{1}{2}\frac{F_z}{m}t^2 \tag{10}$$

where $\vec{s} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\vec{s}_0 = x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}$.

Now, from the above formulas, we see that x,y and z components are independent of one another, in our cases. i.e. when $\vec{F}$ is a constant; we can treat each component separately. For example, the value of x doesn't depend on values such as v_{y0},F_y or F_z.

Finally, a good example. In the presence of gravitation on Earth, the gravitational force F_g is given as follows:

$$\vec{F_g} = -mg\hat{k} \tag{11}$$

where g is approximately given by $9.8m/s^2$. We also have a negative sign in the equation, because the gravitational force is downward. Plugging the this formula to (8),(9) and (10), we get:

$$x = x_0 + v_{x0}t \tag{12}$$

$$y = y_0 + v_{y0}t \tag{13}$$

$$z = z_0 + v_{z0}t - \frac{1}{2}gt^2 \tag{14}$$

Again, we see that x and y components don't get affected by gravitational force. However, we want to note that we have ignored the air friction. In its presence, x, y components are not independent of z component, and the external force is not simply given as F_g as we should have an extra external force due to the friction.

Problem 1. Find the equation of motion (i.e. $x(t)$) for an object with mass m receiving the force $\vec{F} = (4m + 2mt)\hat{i}$ if initial (i.e. $t = 0$) position is at the origin (i.e. $x(0) = 0$) and if initial velocity is 4 (i.e. $\frac{dx}{dt}(0) = 4$.)

Summary

- The acceleration of an object an observer observes doesn't depend on which inertial reference frame the observer is in, as long as he is in an inertial reference frame.