## Newton's law of universal gravitation and Kepler's third law

In our earlier article "Conic sections and Newton's law of gravity," we briefly mentioned that Newton's law of gravity properly explains Kepler's first law. In this article, we will see how it can explain Kepler's third law.

Newton's law of universal gravitation asserts that there are universal attractive force between any two massive bodies. If the mass of each body is  $m_1$  and  $m_2$ , and the distance between them r, the force is given as follows:

$$F = \frac{Gm_1m_2}{r^2} \tag{1}$$

Here G is a proportionality constant called "Newton's constant." Notice also that this is the force that the first object attracts the second object as much as the force that the second object attracts the first object in light of Newton's third law. This force is responsible for objects on Earth such as apples falling downward. On the other hand, the apples attract the Earth by the same amount of force, but this attraction is practically negligible since Earth is much more massive than apples, which make the acceleration of the Earth due to the apples' attraction very tiny compared to the acceleration of the apples due to Earth.

Through this force, the Sun also provides centripetal force for planets to rotate around. If the planet's mass is m and the Sun's mass  $M_{sun}$ , we have:

$$\frac{mv^2}{r} = \frac{GM_{\rm Sun}m}{r^2} \tag{2}$$

where r is the radius of the orbit. Therefore, the rotational velocity v is given by:

$$v = \sqrt{\frac{GM_{\rm Sun}}{r}} \tag{3}$$

Notice that this is independent of the mass of the planet as it cancels out in (2). Then, what would be the period? As the circumference of the orbit is  $2\pi r$ , it takes the following time to rotate once:

$$T = \frac{2\pi r}{v} = 2\pi r \frac{\sqrt{r}}{\sqrt{GM_{\rm Sun}}} \tag{4}$$

Expressed slightly differently, the above formula implies:

$$T^2 = \frac{4\pi^2}{GM_{\rm Sun}}r^3\tag{5}$$

In other words, we see that the period squared is proportional to the radius cubed. In our derivation, we assumed that the orbit was a circle. However, if we derive again considering

the case that the orbit is an ellipse, we obtain that the period squared is proportional to the cube of the semi-major axis (i.e. the mean of the distances farthest and closest to the Sun). This we will not show in this article, but will demonstrate in our later article "Kepler's first and third laws revisited," as this is the original Kepler's third law.

**Problem 1.** Let's say a planet A orbits circularly around the Sun with orbit radius R and period T. If a planet B orbits circularly around the Sun with orbit radius 4R, what would be its period?

## Summary

• The period of a planet T orbiting circularly around the Sun can be obtained as follow:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}, \qquad T = \frac{2\pi r}{v}$$

where m is the mass of the planet, M is the mass of the sun and r is the orbital radius. Notice that the final answer doesn't depend on the mass of the planet.