Newton's third law and the conservation of momentum with calculus

In our earlier article "Newton's third law and the conservation of momentum," we introduced Newton's third law and how it leads to the conservation of momentum. In this article, we will re-visit this issue using calculus and extend the proof of the conservation of momentum to arbitrary multiple numbers of particles.

Momentum of an object with mass m and velocity $\vec{v}$ is defined as follows:

$$\vec{p} = m\vec{v} \tag{1}$$

Given this, observe the following:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} \tag{2}$$

where in the first step we used Newton's second law. Therefore, we conclude:

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{3}$$

Now, we are ready to derive the conservation of momentum. Suppose that we have two objects: object 1 and object 2. Let $\vec{F}_{21}$ be the force the object 1 exerts on the object 2 and $\vec{F}_{12}$ be the force the object 2 exerts on the object 1. If we denote the momentum of the object 1 by p_1, and similarly for the object 2, we have:

$$\vec{F}_{12} = \frac{d\vec{p}_1}{dt} \tag{4}$$

$$\vec{F}_{21} = \frac{d\vec{p}_2}{dt} \tag{5}$$

By Newton's third law, we have $\vec{F}_{21} = -\vec{F}_{12}$. Using this, we obtain:

$$0 = \vec{F}_{12} + \vec{F}_{21} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \frac{d(\vec{p}_1 + \vec{p}_2)}{dt} \tag{6}$$

Therefore, we conclude:

$$\vec{p}_1 + \vec{p}_2 = \text{constant} \tag{7}$$

In other words, we say that the momentum is conserved. All these steps can be re-taken when there are multiple objects, and the conclusion is that the total momentum is conserved as follows:

$$\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n = \text{constant} \tag{8}$$

where n is the number of objects.

For example, when $n = 3$ we have:

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{12} + \vec{F}_{13} \tag{9}$$

$$\frac{d\vec{p}_2}{dt} = \vec{F}_{21} + \vec{F}_{23} \tag{10}$$

$$\frac{d\vec{p}_3}{dt} = \vec{F}_{31} + \vec{F}_{32} \tag{11}$$

$$\begin{aligned} \frac{d(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)}{dt} &= \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{21} + \vec{F}_{23} + \vec{F}_{31} + \vec{F}_{32} \\ &= (\vec{F}_{12} + \vec{F}_{21}) + (\vec{F}_{13} + \vec{F}_{31}) + (\vec{F}_{23} + \vec{F}_{32}) = 0 \end{aligned} \tag{12}$$

Let me conclude this article with some comments. In this article, we had to assume Newton's third law to derive the conservation of momentum. However, there is another method to come to the same conclusion. This is based on symmetry principle, which is more beautiful than Newton's third law. We will learn this in our later article "Noether's theorem."

Summary

- Newton's third law implies the conversation of momentum.