Noether's theorem

Let's say that the Lagrangian is invariant under a specific infinitesimal deformation δq^i. (i.e. $\delta q^i = \epsilon h^i(q)$, where ϵ is an infinitesimal constant, and $h^i(q)$ is a (finite) function of qs.) In particular, this specific deformation doesn't require that the two end points are fixed. Thus, unless the specific deformation is special, the Lagrangian is not guaranteed to be invariant. Then, we have:

$$\begin{aligned} 0 &= \delta L = \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i & (1) \\ &= \left(\frac{\partial L}{\partial q^i} - \partial_t\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right)\delta q^i + \partial_t\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) & (2) \\ &= \partial_t\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) & (3) \end{aligned}$$

where Einstein summation convention is used, and from the second line to the third line, we used the equation of motion. Now, it is apparent that the term in the parenthesis is constant, as its time derivative vanishes. In other words, we found a conserved charge Q, called "Noether charge" as follows:

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta q^i = p_i \delta q^i \tag{4}$$

Now, noticing that $\epsilon Q = p_i \delta q^i$, we can calculate the following quantity:

$$\{q^i, Q\} = \frac{\partial q^i}{\partial q^j}\frac{\partial Q}{\partial p_j} - \frac{\partial q^i}{\partial p_j}\frac{\partial Q}{\partial q^j} = \delta q^i \tag{5}$$

We say "Q generates δq."

The construction so far would be somewhat too abstract without a concrete example. Therefore, let me give you an example. Suppose a system consists of 3 objects with kinetic energy given by the usual one as follows:

$$\frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) + \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2 + \dot{z}_3^2) \tag{6}$$

and the potential energy which depends only on the distances between them as follows:

$$V(|\vec{r}_1 - \vec{r}_2|, |\vec{r}_1 - \vec{r}_3|, |\vec{r}_2 - \vec{r}_3|) \tag{7}$$

These two assumptions are reasonable. The first one is obvious, since that is the definition of kinetic energy. The second is understandable, since all known forces satisfy this property. For example, in Newtonian gravity, the potential energy is inversely proportional to the distances between objects.

Given this, notice that the Lagrangian is invariant under the following transformation:

$$x_1' = x_1 + \epsilon, \qquad x_2' = x_2 + \epsilon, \qquad x_3' = x_3 + \epsilon \tag{8}$$

In other words, in the reference frame S' whose origin is situated at $(x = -\epsilon, y = 0, z = 0)$ in terms of the original reference frame S, must observe the same Lagrangian as the latter. Certainly, S' observes the same kinetic energy as S, since both of them agree on the velocity. Moreover, they also agree on the distance between the objects since S' would observe that the locations of the objects were simultaneously moved by the distance ϵ on the x direction.

Therefore, from (8) we can write:

$$\delta x_1 = \delta x_2 = \delta x_3 = \epsilon \tag{9}$$

and from (4) we have:

$$Q = \epsilon(p_{x1} + p_{x2} + p_{x3}) \tag{10}$$

As ϵ is just an infinitesimal constant, if (10) is conserved, the following is also conserved.

$$Q = p_{x1} + p_{x2} + p_{x3} \tag{11}$$

where we now define Q without the ϵ factor, for conveience. In other words, the sum of the x-component of the momentum of the objects is conserved! (i.e. it doesn't change over time.) All the constructions in this example can be easily generalized to arbitrary number of objects and arbitrary directions (such as y or z) of the momentum. The remarkable thing is that the invariance of the Lagrangian under spatial translation implies the conservation of the total momentum. Since the Lagrangian completely describes a physical system, we can as well say the invariance of the physical system under spatial translation implies the conservation of the total momentum.

Now, let's prove that time translational symmetry implies the conservation of Hamiltonian (i.e. energy). Let's suppose that you move the system by time ϵ, and the action is invariant. Note that I said the action is invariant, rather than the Lagrangian is invariant, because if you change the time, the integration range changes, so it is not guaranteed that the action is invariant even when the Lagrangian is invariant. So, this implies

$$S = \int_{t_1}^{t_2} L(t)dt = \int_{t_1+\epsilon}^{t_2+\epsilon} L(t-\epsilon)dt \tag{12}$$

Remember that Lagrangian is a function of $q^i(t)$ and $\dot{q}^i(t)$. Thus,

$$L(t-\epsilon) = L(q^i(t-\epsilon), \dot{q}^i(t-\epsilon)) \tag{13}$$

Now, notice

$$q^i(t-\epsilon) = q^i(t) - \epsilon\dot{q}^i(t) \tag{14}$$

which implies $\delta q^i = -\epsilon\dot{q}^i$. If we write $L(t-\epsilon) = L(t) + \delta L(t)$, we have

$$\delta L = \partial_t\left(-\frac{\partial L}{\partial\dot{q}^i}\epsilon\dot{q}^i\right) \tag{15}$$

where we used (3). Thus, (12) becomes

$$\begin{aligned}
\int_{t_1}^{t_2} Ldt &= \int_{t_1+\epsilon}^{t_1} Ldt + \int_{t_1}^{t_2} Ldt + \int_{t_2}^{t_2+\epsilon} Ldt + \int_{t_1+\epsilon}^{t_2+\epsilon} \delta Ldt \\
0 &= -\epsilon L(t_1) + \epsilon L(t_2) + \int_{t_1+\epsilon}^{t_2+\epsilon} \partial_t\left(-\frac{\partial L}{\partial\dot{q}^i}\epsilon\dot{q}^i\right) dt & (16)\\
-\epsilon L(t_2) &= -\epsilon L(t_1) - \epsilon\frac{\partial L}{\partial\dot{q}^i}\dot{q}^i\Big|_{t_1+\epsilon}^{t_2+\epsilon} & (17)\\
-L(t_2) &= -L(t_1) - \frac{\partial L}{\partial\dot{q}^i}\dot{q}^i\Big|_{t_1+\epsilon}^{t_2+\epsilon} & (18)
\end{aligned}$$

Since we are sending the limit $\epsilon \to 0$, we have

$$-L(t_2) + \frac{\partial L}{\partial\dot{q}^i}\dot{q}^i(t_2) = -L(t_1) + \frac{\partial L}{\partial\dot{q}^i}\dot{q}^i(t_1) \tag{19}$$

Indeed, this is exactly the conservation of Hamiltonian! i.e. $H(t_2) = H(t_1)$

If a physical system has some invariance of action under a certain deformation, we say that it has a symmetry. Thus, we can say, for every symmetry, there is always a conserved charge. As other examples, we will later see that the fact that physics doesn't depend on the phase of wave function (i.e. invariant under local gauge transformation) implies the conservation of electric charge.

Now, let's re-write (4) and (5) slightly differently. We know that Q deforms q. In other words, applying ϵQ changes q. (Now, we are using Q in (11), i.e., the one defined without ϵ factor.) If we successively apply Q with bigger ϵs, q will change more and more. Let's say that we parametrize this deformation of q by a parameter a, and applying ϵQ changes the parameter by ϵ. In other words,

$$\{q^i(a), \epsilon Q\} = \delta q^i = q^i(a+\epsilon) - q^i(a) \tag{20}$$

Then,

$$\begin{aligned}
\epsilon\{q^i(a), Q\} &= \epsilon\frac{dq^i}{da} \\
\{q^i(a), Q\} &= \frac{dq^i}{da} & (21)
\end{aligned}$$

Given this, let us give you another example. In an earlier article, we have seen that the Hamiltonian H generates time translation as follows.

$$\{f, H\} = \frac{df}{dt} \tag{22}$$

Comparing this with (21), we see that the parameter a there is the time t.

Problem 1. Let's say we use the notation of (21). Then, prove following.

$$f(a) = f_0 + a\{f, Q\}_0 + \frac{a^2}{2!}\{\{f, Q\}, Q\}_0 + \frac{a^3}{3!}\{\{\{f, Q\}, Q\}, Q\}_0 + \cdots \tag{23}$$

where the subscript 0 denotes the value evaluated when $a = 0$.

Problem 2. Evaluate the above expression when $f = x$ (i.e. the x coordinate of position) and $Q = L_z$ (i.e. the z component of angular momentum). Repeat the exercise when $f = y$ (i.e. the y coordinate of position) and $Q = L_z$, and thereby show that L_z generates the (clockwise) rotation around x-y plane (for $a > 0$), and a in this case is given by the rotation angle.

Finally, let me argue why God made Newton's third law. Let's say you set a laboratory in Daejeon and perform physics experiments. Then you will get certain results. Suppose you perform the same physics experiments with the same conditions in certain other locations. Then, you will get the same results. This means that there is space translation symmetry, and as the consequence, the momentum conserves. Suppose you perform physics experiments today, and perform the same physics experiments again with the same condition ten days later. Then, you will get the same result. This means that there is time translation symmetry, and as the consequence, the energy conserves. I think, God would not want us to perform the same experiments different place and different time and not get the same results, because we would not be able to find the law of physics then. Thus, momentum and energy must be conserved. But, for the momentum to be conserved, we need Newton's third law. I think that is the reason why God made Newton's third law.

Summary

- For every symmetry, there is a conserved Noether charge.
- Noether charge is given by $Q = p_i \delta q^i$. It generates δq^i.
- The Noether charge for the time translation is energy and the Noether charge for the space translation is momentum.