Non-Euclidean geometry

In our earlier article "Manifold," we introduced the concept of manifold. There, we stated that $\mathbb{R}^n$ was the Euclidean space. Precisely speaking, this was not a correct statement. Let me give you the correct statement. If you take two vectors in the Euclidean space $\mathbb{R}^n$, say $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$ then the inner product is given by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \tag{1}$$

Notice that if $\vec{z}$ is a non-zero vector, then

$$\vec{z} \cdot \vec{z} > 0 \tag{2}$$

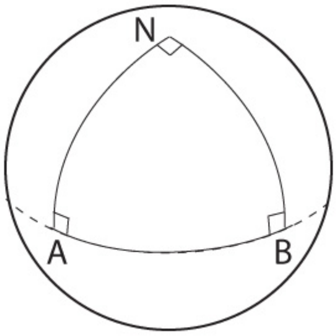
The Euclidean space always admits the coordinate system which satisfy (1). Such a coordinate system is called the Cartesian coordinate system which you are familiar with. I say "admit" because there are other coordinate systems, which do not satisfy (1), but you can use to describe the same Euclidean space. Polar coordinate and spherical coordinate are good examples. Of course, even though (1) is re-defined in the ways appropriate for the polar coordinate and the spherical coordinate, (2) is always satisfied.

Now, let's talk about the Minkowski space. The Minkowski space (sometimes called "the Lorentzian space") is also given by $\mathbb{R}^n$, but the dot product of two vectors, say $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$ is given by

$$\vec{x} \cdot \vec{y} = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \tag{3}$$

If you read our earlier article "Rotation and the Lorentz transformation, orthogonal and unitary matrices," it should be clear why dot product must be defined this way in Minkowski space. Notice also that (2) is not always satisfied. Again, if a space admits the coordinate system which satisfy (3), it's the Minkowski space.

The Euclidean space and the Minkowski space are examples of manifolds with no curvature. They are not curved at all. A good example of curved manifold would be spheres. Let's first look at S^2 (i.e. 2-sphere, or simply sphere) closely, and what the rules of the geometry look like on 2-sphere. This is an example of what is called "non-Euclidean geometry" which we briefly mentioned in "Curved Space." Certainly 2-sphere is not an Euclidean space.



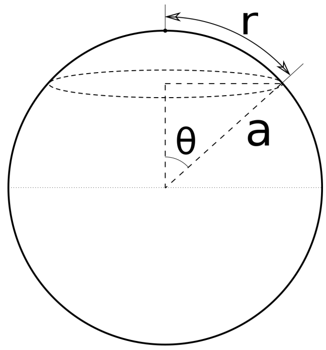


Figure 1: the sum of the angles of a triangle on a sphere is always bigger than 180°.

Figure 2: the circumference to radius ratio of a circle on a sphere is always less than 2π.

In "Curved Space," we mentioned some properties of non-Euclidean space, such as the sum of the angles of the triangle not being 180°, and failure to satisfy Euclid's parallel postulate and so on. See Fig. 1.

Here, I want to mention another property of non-Euclidean space. The ratio of the circumference of a circle to radius fails to be 2π. For example, on a sphere, the ratio of the circumference of a circle to the radius is smaller than 2π. As we treated a triangle on a sphere in Fig. 1, we will treat a circle on a sphere in Fig. 2. In Fig. 2, the center of the circle is the North Pole. Just like in the Euclidean case, a circle of radius r is defined by the set of points distance r away from the center. Let's calculate the circumference. The angle θ in the figure is given by $\theta = r/a$. Thus, the circumference is given by

$$C = 2\pi a \sin\theta = 2\pi a \sin\frac{r}{a} \tag{4}$$

Then, the ratio is given by

$$\frac{C}{r} = 2\pi \frac{\sin\frac{r}{a}}{\frac{r}{a}} \tag{5}$$

As $\frac{\sin\theta}{\theta} < 1$, we see that the ratio is smaller than 2π. Also, it is interesting that there is a maximum value for the circumference. It is $2\pi a$. See Fig. 3. This is achieved when $\frac{r}{a} = \pi/2$. That is $r = \pi a/2$. Notice also that the circumference decreases as r increases if r is bigger than $\pi a/2$. What is the area of a circle with radius r? See Fig. 4. The area of "stripe" is given by $C\,dr$. We need to integrate this as follows.

$$A = \int_0^r 2\pi a \sin\frac{r}{a} dr = 2\pi a^2 (1 - \cos\frac{r}{a}) \tag{6}$$

Similarly as before, this value is smaller than $4\pi r^2$. (**Problem 1.** What is the maximum A can have? This should be equal to the surface area of 2-sphere with radius a. Thus, check that your answer is correct.)

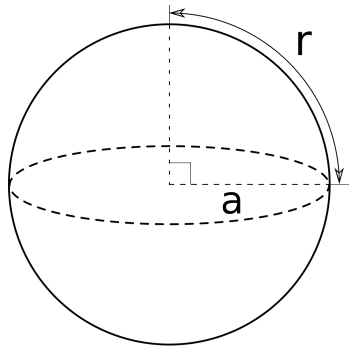


Figure 3: the biggest circle on a sphere

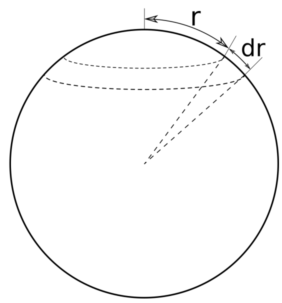


Figure 4: the area of a circle on a sphere

If a is bigger, it means that the sphere is less curved. Think of the Earth. As a is very big, we don't usually notice that the Earth is not flat, but actually round. So, when a is big, (4) and (6) should reduce to our usual flat formula $C = 2\pi r$, and $A = 4\pi r^2$. (**Problem 2.** Show this using Taylor series!)

Having talked about 2-sphere, let's talk about 3-sphere. Let's say again the radius of the 3-sphere is a. Then, a circle in this 3-sphere satisfies (4). What would be the surface area of a 2-sphere in the 3-sphere? It turns out[1]

$$A = 4\pi \left(a \sin \frac{r}{a} \right)^2 \tag{7}$$

An easy way of interpreting this formula is that you regard $a \sin \frac{r}{a}$ as the "effective" radius as in (4). In other words, 2π times the effective radius is the circumference and 4π times the square of the effective radius is the area of the sphere.

Problem 3. Calculate the volume of 3-ball of radius r inside 3-sphere using (7). Then, show that this reduces to the familiar $V = \frac{4}{3}\pi r^3$ in the limit a goes to infinity. What is the maximum volume a 3-ball can have?

Spheres are examples of constant curvature. Curvature denotes how much manifolds are curved, and because of the symmetry of spheres all the points on the sphere have the same curvature. The curvature turns out to be $1/a^2$ in our cases. Therefore, precisely speaking, spheres are examples of constant positive curvature.

There are examples of constant negative curvature. They are called "pseudo-spheres." For the pseudo-spheres with constant curvature $-1/a^2$, if we actually calculate the analogous formula for the formulas we had for spheres, all the sine functions in our formulas are replaced by sine hyperbolic

[1] We will later show this in our article "The FRW metric and the Friedmann equations."

functions. For example, for (4), we have

$$C = 2\pi a \sinh \frac{r}{a} \tag{8}$$

for (7), we have

$$A = 4\pi \left(a \sinh \frac{r}{a} \right)^2 \tag{9}$$

May I suggest an "illegal" way to derive these two equations? Mathematicians will be aghast at this derivation for its lack of rigor, but I like putting it in this way. As we mentioned, a sphere has a constant curvature of $1/a^2$. Recalling that a pseudo-sphere has a constant curvature of $-1/a^2$, we can regard a pseudo-sphere as a sphere with radius "ia," as its curvature would be $-1/a^2$. Then, (4) becomes

$$C = 2\pi(ia) \sin \left(\frac{r}{ia} \right) \tag{10}$$

which is exactly (8). The derivation of (9) is also straightforward.

Also, even though we will not show here, it turns out that there are infinitely many lines that are parallel to a certain line and pass a point in this pseudo-sphere. Also, the sum of the angles of the triangle is always less than 180°.

Problem 4. Show that the ratio of the circumference to the radius is always bigger than 2π for circles in 2 pseudo-sphere. Also show that in the flat limit (i.e. a goes to infinity), the ratio becomes 2π. Notice that in this limit, the curvature (i.e. $-1/a^2$) approaches 0.

Problem 5. Show that (9) can be as big as it can be. This shows that a 2 pseudo-sphere has an infinite area.

Let me conclude this article with a comment. In "Cosmological principle and my view on philosophy" I introduced the cosmological principle. Cosmological principle suggests that our Universe has a constant curvature. So, there are three possibilities for the geometry of the space of our Universe. 3-sphere, the Euclidean space $\mathbb{R}^3$ and 3-pseudo sphere. If our universe is 3-sphere, the size of our universe is finite. If our Universe is the Euclidean space or 3-pseudo sphere, the size of our universe is infinite. All these three possibilities for our Universe have been considered by cosmologists. We will talk more about these three possibilities in our later article "The FRW metric, and the Friedmann equation."

Summary

- The inner product of $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$, two vectors in the Euclidean space $\mathbb{R}^n$, is given by $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

- The inner product of $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$, two vectors in the Minkowski space $\mathbb{R}^n$, is given by $\vec{x} \cdot \vec{y} = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

- The Euclidean space and the Minkowski space are examples of manifolds with no curvature.

- In the limit, the radius of a sphere becomes infinite, (i.e. the curvature of a sphere becomes zero) the sphere becomes a flat space.

- A pseudo-sphere has a constant negative curvature.

- In the limit, the curvature of a pseudo-sphere becomes zero, it becomes a flat space.