Partial Differential Equations

Partial differential equations(PDE) are differential equations that involve partial derivatives. Here is an example that open shows up in physics:

$$\frac{\partial^2 h}{\partial t^2} = v^2 \frac{\partial^2 h}{\partial x^2} \tag{1}$$

It's actually called "(one-dimensional) wave equation." The solution is given by:

$$h = f(x - vt) + g(x + vt) \tag{2}$$

One can check that this is indeed a solution by following way. Let, $w \equiv x - vt$. Then,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial w} \tag{3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial w}) = \frac{\partial w}{\partial x}(\frac{\partial^2 f}{\partial w^2}) = \frac{\partial^2 f}{\partial w^2} \tag{4}$$

Similarly, one can check:

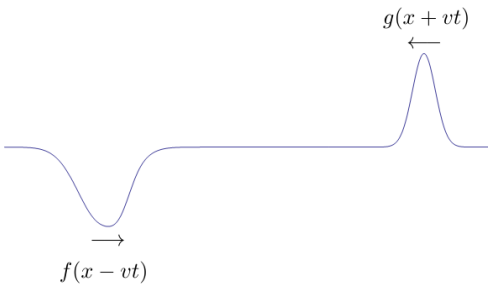
$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial w^2} \tag{5}$$

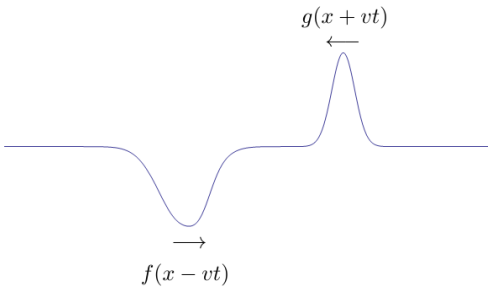
Therefore, $h = f(x - vt)$ satisfies (1). Similarly, one can also show that $h = g(x + vt)$ satisfies (1). Then, their sum also satisfies (1) since:

$$\frac{\partial^2 (f(x - vt) + g(x + vt))}{\partial t^2} = \frac{\partial^2 (f(x - vt))}{\partial t^2} + \frac{\partial^2 (g(x + vt))}{\partial t^2} \tag{6}$$

$$v^2 \left(\frac{\partial^2 (f(x - vt) + g(x + vt))}{\partial x^2} \right) = v^2 \left(\frac{\partial^2 (f(x - vt))}{\partial x^2} + \frac{\partial^2 (g(x + vt))}{\partial x^2} \right) \tag{7}$$

Now, some interpretations regarding the solution (2). See the figures. You see that $f(x - vt)$ is moving right while $g(x + vt)$ is moving left. Also, both of them don't change the shape as they move on. So, this certainly is a equation that describes moving waves. Moreover, after they pass through each other, they go on their way preserving their shape as if nothing happened, since the solution f and g are completely separate. Actually, if you study string theory, you will encounter the equation (1), and you will learn that its solution is given by (2). String theorists call $f(x - vt)$ right-moving wave and $g(x + vt)$ left-moving wave for obvious reasons.


$g(x+vt)$
$f(x-vt)$


$g(x+vt)$
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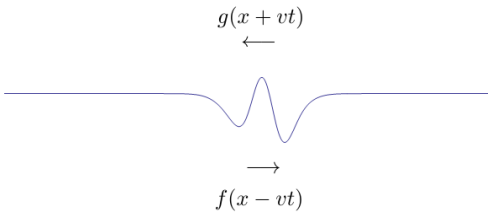
$g(x+vt)$
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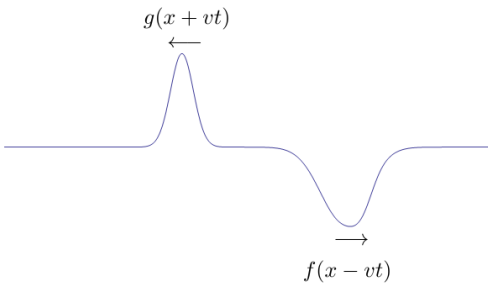
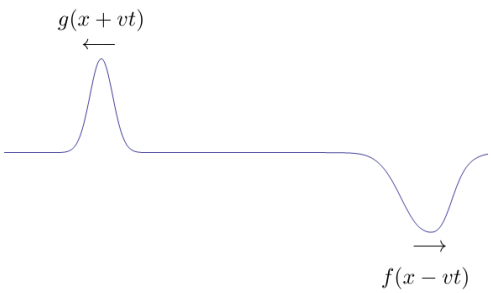

$g(x+vt)$
$f(x-vt)$


$g(x+vt)$
$f(x-vt)$

Summary

- Partial differential equations(PDE) are differential equations that involve partial derivatives.
- The solution to
$$\frac{\partial^2 h}{\partial t^2} = v^2 \frac{\partial^2 h}{\partial x^2}$$
is given by
$$h = f(x - vt) + g(x + vt)$$