The Poisson bracket

The Poisson bracket of two quantities f and g is defined as follows:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \tag{1}$$

One of the properties of Poisson bracket is that it is anti-symmetric. In other words,

$$\{f, g\} = -\{g, f\} \tag{2}$$

It is left as an exercise to readers to verify this, as it is obvious from the definition of the Poisson bracket. This also implies:

$$\{f, f\} = -\{f, f\} \tag{3}$$

which, in turn, implies:

$$\{f, f\} = 0 \tag{4}$$

Now, let's calculate the following quantity:

$$\begin{aligned} \{f, H\} &= \sum_i \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i} && (5)\\ &= \sum_i \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial p_i}\dot{p}_i && (6)\\ &= \frac{df}{dt} && (7) \end{aligned}$$

$$\{f, H\} = \frac{df}{dt} \tag{8}$$

Here, we see that Hamiltonian is closely related to the "time translation," as physicists call it. $(f \to f + \frac{df}{dt}\delta t)$.

In the last article, we showed that Hamiltonian is conserved, i.e., remains constant as time goes on. Now, let's derive this fact again, which seems a bit more elegant. By plugging $f = H$ in (4) and (8), we obtain:

$$0 = \{H, H\} = \frac{dH}{dt} \tag{9}$$

There are a couple of more crucial properties of Poisson bracket. As an exercise, the readers should verify the following:

$$\{q^i, p_j\} = \delta^i_j \tag{10}$$

and, the following:

$$\{A, B + C\} = \{A, B\} + \{A, C\} \tag{11}$$

$$\{A, BC\} = B\{A, C\} + \{A, B\}C \tag{12}$$

which can be derived straightforwardly from the definition of Poisson bracket using Leibniz's rule. (i.e. $(fg)' = f'g + fg'$)

Problem 1. Prove the following formula.

$$f(t) = f_0 + \{f, H\}_0 + \frac{t^2}{2!}\{\{f, H\}, H\}_0 + \frac{t^3}{3!}\{\{\{f, H\}, H\}, H\}_0 + \cdots \tag{13}$$

where the subscript 0 denotes the value evaluated when $t = 0$.

Problem 2. Let's denote the position of an object in Cartesian coordinate by $x = q^1, y = q^2, z = q^3$ and its momentum by $p_x = p_1, p_y = p_2, p_z = p_3$. Then (10) implies $\{x, p_x\} = \{y, p_y\} = \{z, p_z\} = 1$ and all other Poisson brackets vanish. Using this relation, along with $L_z = xp_y - yp_x$ (i.e. the zth component of angular momentum), calculate the following. The answer to this problem will turn out to be useful in solving a problem in our later article "Noether's theorem." (Hint[1])

$$\{x, L_z\} =?, \qquad \{y, L_z\} =?, \qquad \{z, L_z\} =? \tag{14}$$

Summary

- $\{f, g\} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$
- $\{f, g\} = -\{g, f\}$ which implies $\{f, f\} = 0$.
- $\{f, H\} = \frac{df}{dt}$
- The Hamiltonian is conserved as $\{H, H\} = \frac{dH}{dt} = 0$.
- $\{q^i, p_j\} = \delta^i_j$.
- $\{A, B + C\} = \{A, B\} + \{A, C\}$
- $\{A, BC\} = B\{A, C\} + \{A, B\}C$

[1] Use (11) and (12).