Poisson's equation

Recall that the electric field satisfies the Gauss law as follows:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

If we plug in $\vec{E} = -\nabla\phi$ to the above equation, we get the following relation between the electric charge and the electric potential.

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{2}$$

This equation is called "Poisson's equation" for electrostatics. Any equation in the form of

$$\nabla^2\phi(x, y, z) = f(x, y, z) \tag{3}$$

is called "Poisson's equation."

We also know that the gravity follows inverse square law as Coulomb force. So, it is right to suspect that there is also Poisson's equation for gravity. Let's find it.

To this end, let's first introduce gravitational potential Φ. If an object has mass m, and it is at a certain position $\vec{r}$, gravitational potential Φ is defined as follows:

$$\Phi(\vec{r}) = \frac{U(\vec{r})}{m} \tag{4}$$

where $U(\vec{r})$ is the gravitational potential energy at position $\vec{r}$. As U is proportional to m, Φ is independent of m. For example, if masses M_is are situated at $\vec{r}_i$, we have:

$$\Phi(\vec{r}) = -\sum_i \frac{GM_i}{|\vec{r}_i - \vec{r}|} \tag{5}$$

Now, what is the acceleration $\vec{g}$ due to this gravitational potential? We can obtain it as follows:

$$\begin{aligned} \vec{F} &= -\nabla U \\ m\vec{g} &= -m\nabla\Phi \\ \vec{g} &= -\nabla\Phi \end{aligned} \tag{6}$$

We also know that $\vec{g}$ satisfies the inverse square law as follows:

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \tag{7}$$

Given this, recall that the following inverse square Coulomb's law

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{8}$$

implies (1). Therefore, if one is careful about the factors, (7) must imply the following:

$$\nabla \cdot \vec{g} = -4\pi G\rho \tag{9}$$

where ρ is the mass density. Using (6), we conclude:

$$\nabla^2 \Phi = 4\pi G\rho \tag{10}$$

This equation is equivalent to Newton's law of gravity, as much as Gauss's law is equivalent to Coulomb's law. This is Poisson's equation for gravity.

Summary

- Poisson's equation is given by $\nabla^2\phi$ =something, where something is the electric charge density with some factors when ϕ is electric potential, and something is the mass density with some factors when ϕ is the gravitational potential.