Totally inelastic relativistic collision

In this article, we will treat totally inelastic collision relativistically. Suppose an object with a rest mass m with initial speed v totally inelastically collides another object with a rest mass m, initially at rest. As they are colliding inelastically they will stick completely together and move as one body. Then, what is their final speed? We know that the answer is $v/2$ non-relativistically. But, this answer must change if we consider the relativistic effect.

Let's calculate the total energy before the collision. It is given by $\gamma mc^2 + mc^2 = (\gamma+1)mc^2$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. The total energy after the collision is same because of the energy conservation. Let's calculate the total momentum before the collision. It is given by $\gamma mv + 0 = \gamma mv$. The total momentum after the collision is same because of the momentum conservation.

The two objects have now become one body. Let's say that the rest mass of this body is M, and its speed v_f. Then, the energy of this body is given by $\gamma_f Mc^2$ where $\gamma_f = 1/\sqrt{1 - v_f^2/c^2}$, and the momentum $\gamma_f M v_f$.

However, from the energy-momentum conservation, we have

$$(\gamma + 1)mc^2 = \gamma_f Mc^2 \tag{1}$$

$$\gamma mv = \gamma_f M v_f \tag{2}$$

Dividing (2) by (1), we obtain

$$\frac{\gamma v}{(\gamma + 1)c^2} = \frac{v_f}{c^2} \tag{3}$$

Thus, we conclude

$$v_f = \frac{\gamma}{\gamma + 1} v \tag{4}$$

Problem 1. Show that (4) reduces to the familiar $v_f = v/2$ in the non-relativistic limit.

What would be the rest mass of the combined body? Would it be $2m$? Let's check whether it is $2m$ by explicitly calculating M. If we let the energy of the combined body E_f, and the momentum p_f, we know

$$E_f^2 - p_f^2 c^2 = M^2 c^4 \tag{5}$$

Remembering that E_f is given by (1) and p_f by (2), we get

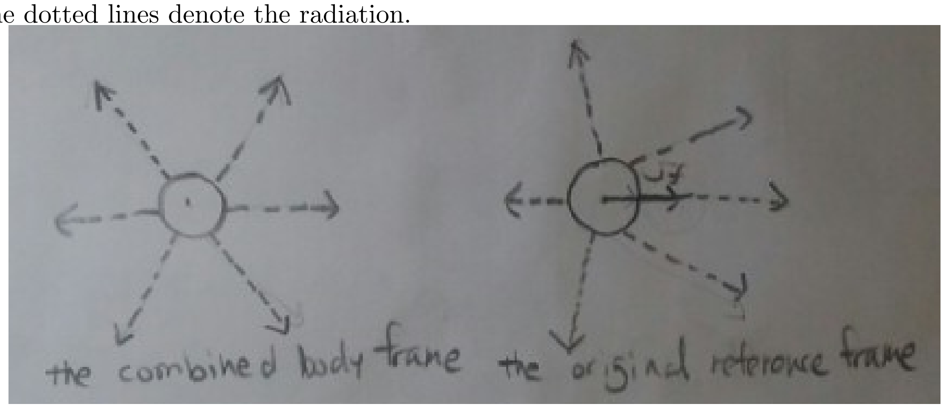
$$\left((\gamma + 1)^2 - (\gamma v/c)^2\right) m^2 = M^2 \tag{6}$$

Using $\gamma^2(1 - v^2/c^2) = 1$, we get

$$M = \sqrt{2 + 2\gamma}\, m \tag{7}$$

Thus, M is *not* equal to $2m$. It is bigger than $2m$.[1] This caused some confusions among some of my classmates at Korean Physics Olympiad camp, so they asked teaching assistants.[2] The answer was that the combined final body has more rest mass than the sum of the initial ones, as the former has the heat generated by the inelastic collision. Remember that heat is energy and energy is mass.

However, then, another question arises. If the combined final body loses heat by radiation, it will lose its rest mass as well; the rest mass will be eventually $2m$. If you see this in the combined final body frame, the velocity of the combined final body will not change during radiation. It will be zero both before and after; there is no reason that the direction of radiated light will favor one way than the others. This implies in the original reference frame, the velocity of the combined final body won't change; it will be v_f both before and after. However, its momentum will change from $\gamma_f M v_f$ to $\gamma_f 2m v_f$. Then, the problem is: how can the combined body lose its momentum? Isn't the radiation of heat isotropic? ("Isotropic" means equal in all directions.) So, we asked the teaching assistant again, and he explained that in the original reference frame the radiation of the heat is not isotropic. See the figure. This should be familiar if you read our earlier article "Mass-energy equivalence." The dotted lines denote the radiation.



Actually several years later, I gave as an exercise checking that the radiation loses the right amount of momentum to the Korean team members for International Physics Olympiad, when I was a teaching assistant. One of them got the right answer.

Problem 1. In this problem, we will consider the collision of a pion(π) and a nucleon. (Nucleon is the collective term for protons and neutrons, the constituents of nuclei.) A moving pion collides with a nucleon at rest, and they become a Delta(Δ) particle. If the mass of Delta particle is 1230 MeV/c^2, the mass of pion 140 MeV/c^2, and the mass of nucleon 940 MeV/c^2 (the masses of proton and neutron are almost same), what should be the kinetic energy of pion to produce a Delta particle? (Hint[3]) We will talk about this phenomenon in our later article on quantum field theory.

[1] Remember that for a non-zero v, $\gamma > 1$.

[2] Their starting point of the question is not exactly same as this, but I presented the matter in this way for clarity. In any case, they asked why the rest mass is not conserved during collisions.

[3] Write an analogous formula for (6). Then, solve for γ using the fact $\gamma^2(1 - v^2/c^2) = 1$.

Summary

- During relativistic collision, the rest mass is not necessarily conserved, but the total relativistic energy is conserved.