Quantum corrections to Hawking radiation spectrum

As advertised in our earlier articles "Discrete area spectrum and the Hawking radiation spectrum II" and "The Bose-Einstein distribution, the Fermi-Dirac distribution and the Maxwell-distribution," let me explain how my own research on Hawking radiation was related to the latter article. I will closely follow my original paper "Quantum corrections to Hawking radiation spectrum."

1 Selection rules for quantum black holes

In this section, we will rephrase our explanations in "Discrete area spectrum and the Hawking radiation spectrum II" using mathematical formulas to fix the notation.

Let's say that we have the following area eigenvalues (i.e. the unit areas):

$$A_i = A_1, A_2, A_3, A_4, A_5, A_6.... \tag{1}$$

Then, the black hole area A must be given by the following formula:

$$A = \sum_i N^i A_i \tag{2}$$

where the N^is are non-negative integers. Here, we can regard the black hole as having $\sum N^i$ partitions, each of which has one of the A_i as its area. In this mathematical language, we can express the consideration of Barreira, Carfora and Rovelli as follows: the black hole with initial area $A_{\rm int} = \sum N^i_{\rm int} A_i$ can turn into a black hole with final area $A_{\rm fin} = \sum N^i_{\rm fin} A_i$ through the emission of photons, as long as $A_{\rm fin} < A_{\rm int}$, without any restrictions on the set of $N^i_{\rm fin}$.

However, if we assume that the emission of a photon is local, this is not the case. For a photon to be emitted locally, it should be emitted from a single area quantum, not simultaneously from multiple area quanta separated in space. Possibly following these considerations, Krasnov argued that[1]

> "Consider a quantum process in which the black hole jumps from a state $|\Gamma\rangle$ to state $|\Gamma'\rangle$, such that the horizon area changes. This, for example, can be a process in which one of the flux lines piercing the horizon breaks, with one of the ends falling into the black hole and the other escaping to infinity (see Fig. 1b). This is

[1] K. V. Krasnov, "Quantum geometry and thermal radiation from black holes," Class. Quant. Grav. **16**, 563 (1999) [gr-qc/9710006].

> an example of the emission process; the two ends of the flux line can be thought of as the two particle anti-particle quanta in Hawking's original picture [6] of the black hole evaporation"

Translating this into a mathematical formula, what Krasnov argues is the following:

$$\Delta A = A_j - A_i \tag{3}$$

for some $A_i > A_j$. In other words, the partition with area A_i on the black hole horizon shrinks into a partition with area A_j upon the emission of a particle because the anti-particle reaches this partition of the black hole horizon.

However, Krasnov's argument is also troublesome. In section 4, we will explain why the selection rule should be

$$\Delta A = -A_i \tag{4}$$

for some i. Before doing so, we will explain the consequences of (4) in the next two sections.

2 The discreteness of the Hawking radiation spectrum

In our later article "A Relatively Short Introduction to General Relativity," we will see the following, in natural units ($G = c = \hbar = 1$):

$$r = 2M \tag{5}$$

$$A = 4\pi r^2 = 16\pi M^2 \tag{6}$$

$$kT = \frac{1}{8\pi M} \tag{7}$$

where A is the horizon area of the black hole, T its temperature, r its radius, and M its mass, and k is Boltzmann's constant. Here, we consider the case of a Schwarzschild black hole for simplicity, but it can easily be generalized to the generic case as is done in section 3.

Now consider the emission of a photon from the black hole. As the photon is emitted, the black hole loses energy; thus, its area decreases by A_i, the unit area predicted by loop quantum gravity as we argued in (4) in the last section. From this consideration, we can calculate E_{photon}, the energy of the emitted photon. First of all, the mass of the black hole decreases as

$$\Delta M = -E_{photon} \tag{8}$$

Then, considering (6) and (7), the area of the black hole decreases as

$$\Delta A = 32\pi M \Delta M = -32\pi M E_{photon} = -\frac{4E_{photon}}{kT} = -A_i \tag{9}$$

where in the last step, we assert that the black hole area must be decreased by the unit area A_i predicted by loop quantum gravity. Therefore, we conclude the following:

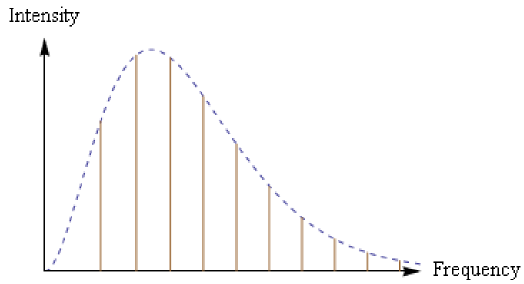

Figure 1: Isolated horizon framework

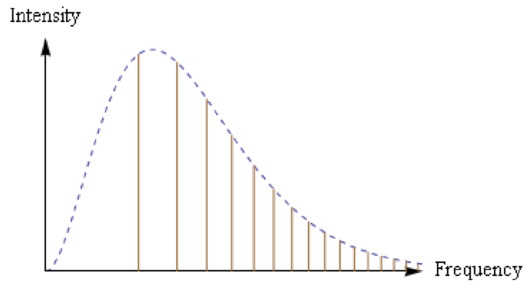

Figure 2: Tanaka-Tamaki scenario

$$E_{photon} = \frac{A_i}{4} kT \tag{10}$$

Here, we see easily that the energy of the emitted photon is quantized because A_i is quantized. In particular, as loop quantum gravity predicts that a non-zero minimum area exists, a non-zero energy exists for the photons emitted from a black hole of a given temperature.

In my original paper "Quantum corrections to Hawking radiation spectrum," I considered three then available scenarios for the area spectrum in loop quantum gravity.

In the case of the isolated horizon framework, the minimum area is given by $4\pi\sqrt{3}\gamma$ where γ is the Immirzi parameter. Therefore, we have the following for the minimum energy of the emitted photon:

$$E_{min} \approx 1.49kT \tag{11}$$

(see Fig. 1). The Hawking radiation is truncated below this energy. The discrete frequency values allowed for Hawking radiation are represented by solid lines. In the case of the Tanaka-Tamaki scenario, the minimum area is given by $4\pi\gamma$, where γ is the Immirzi parameter for this case. This gives the following for the minimum energy of emitted photon:

$$E_{min} \approx 2.462kT \tag{12}$$

(see Fig. 2). In the case of the Kong-Yoon scenario, the minimum area is given by $4\pi\sqrt{2}$. Therefore, we have the following minimum energy:

$$E_{min} \approx 4.44kT \tag{13}$$

(see Fig. 3).

3 Alternative derivation

In this section, we present a simpler derivation. From thermodynamics, we have the following:

$$\Delta Q = T\Delta S \tag{14}$$

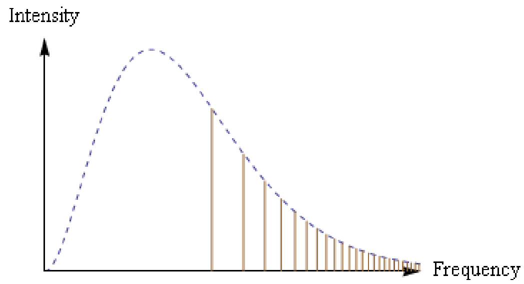

Figure 3: Kong-Yoon scenario

Plugging in the equalities

$$\Delta Q = -E_{photon} \tag{15}$$

$$\Delta S = -kA_i/4 \tag{16}$$

we recover (10).

4 Griffiths' quantum mechanics and Pathria's statistical mechanics

In his famous textbook *Introduction to Quantum Mechanics*, Griffiths considers a statistical mechanics problem as follows:

> "Now consider an arbitrary potential, for which the one-particle energies are $E_1, E_2, E_3, \cdots$, with degeneracies $d_1, d_2, d_3, \cdots$. Suppose we put N particles into this potential; we are interested in the configuration $(N_1, N_2, N_3, \cdots)$, for which there are N_1 particles with energy E_1, N_2 particles with energy E_2 and so on. How many different ways can this be achieved?"

Then, he shows that the answer is given by the following for the case of bosons:

$$Q = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!} \tag{17}$$

He also explains that we have the following two conditions:

$$\sum_{n=1}^{\infty} N_n = N, \qquad \sum_{n=1}^{\infty} N_n E_n = E \tag{18}$$

The first condition requires that the total number of particles is N while the second requires that the total energy is E. To find the most probable configuration $(N_1, N_2, N_3, \cdots)$, he maximizes $\ln Q$ as follows:

$$G \equiv \ln Q + \alpha \left[N - \sum_{n=1}^{\infty} N_n \right] + \beta \left[E - \sum_{n=1}^{\infty} N_n E_n \right] \tag{19}$$

where G is to be maximized and α and β are Lagrange multipliers. He concludes that

$$N_n = \frac{d_n}{e^{\alpha+\beta E_n} - 1} \tag{20}$$

Of course, in the case of photons, the number N is not conserved, so we set $\alpha = 0$ in (19) and (20). Furthermore, we know $\beta = 1/(kT)$, which implies

$$N_n = \frac{d_n}{e^{E_n/(kT)} - 1} \tag{21}$$

We also know (see, e.g., Section 6.4 of "Statistical Mechanics" by Pathria) that the intensity $I(E_n)$ of photons emitted through black body radiation is given by

$$I(E_n) = \frac{c}{4} N_n A = \frac{c}{4} \frac{d_n}{e^{E_n/(kT)} - 1} A = \frac{c}{4} \frac{8\pi f^2 df}{e^{hf/(kT)} - 1} A \tag{22}$$

In the last two expressions, we have substituted the density of states for the degeneracy d_n in the numerator and written the photon energy E_n in terms of the frequency f as hf. Recalling that the black hole (or any black body) loses energy hf upon emission of a photon with frequency f, we can write

$$\Delta E = -hf \tag{23}$$

thus,

$$\Delta E = -E_n \tag{24}$$

This equation shows that only the radiation associated with E_n (i.e., a single area quanta or nth unit area) is possible. This can be seen better by noticing that the second equation of (18) runs parallel with (2). They are actually related by (10). Therefore, we derived (4) (i.e., $\Delta A = -A_i$).

Now, suppose a hypothetical case in which the area deduction is given by $\Delta A = A_j - A_i$ as Krasnov argued. In such a case, we would have $\Delta E = E_j - E_i$, which implies that the energy of the emitted photon is given by $hf = E_i - E_j$. Given this, let's compare the black body radiation formula in this hypothetical case with (21). The denominator does not match as (21)'s denominator is $e^{E_n/(kT)} - 1$ while Krasnov's hypothetical one would be $e^{(E_i - E_j)/(kT)} - 1$. They are clearly different. Furthermore, the numerator does not match either. In the case of (21), we have the degeneracy of the nth quanta given as d_n. In Krasnov's hypothetical case, whether the degeneracy should be d_i or d_j or $d_i d_j$ is not clear. Perhaps no consistent way exists to assign a value to the numerator such that it reduces to d_n in the case where $E_i = E_n$ and $E_j = 0$ but is different from d_n when $E_i = E_n$ but $E_j \neq 0$. In conclusion, Krasnov's area deduction condition is wrong as it cannot reproduce (21).

Summary

- $\Delta A = -A_i$. (Upon emission of a single photon during Hawking radiation, only a single area quantum can completely disappear.)