Taylor series

Taylor series can be easily learned by students who know some basic calculus. Taylor series (or Taylor expansion) is a way of representing a function in terms of an infinite power series.

Let's consider the following case. If y is sufficiently close to x, you may approximate $f(y)$ as $f(x) + f'(x)(y - x)$. A natural question you may want to ask is: Is there any better way to approximate this function?

Given this situation, you may write $f(y)$ as the following.

$$f(y) = f(x) + f'(x)(y - x) + A_2(y - x)^2 + A_3(y - x)^3 + A_4(y - x)^4 \cdots \tag{1}$$

This certainly seems to be a better approximation, if we could determine A_2, A_3, A_4 and so on.

Let's differentiate (1)

Then we get

$$f'(y) = 1 \cdot f'(x) + 2 \cdot A_2(y - x) + 3 \cdot A_3(y - x)^2 + 4 \cdot A_4(y - x)^3 \cdots \tag{2}$$

Differentiating again, we get

$$f''(y) = 2 \cdot 1 \cdot A_2 + 3 \cdot 2 \cdot A_3(y - x) + 4 \cdot 3 \cdot A_4(y - x)^2 \cdots \tag{3}$$

As this formula must be satisfied for y near x, let's plug in $y = x$. Then, we get

$$f''(x) = 2 \cdot 1 \cdot A_2 + 0 + 0 + 0 \cdots = 2 \cdot 1 \cdot A_2 \tag{4}$$

So, we get the following relationship.

$$A_2 = f''(x)/2! \tag{5}$$

Also, differentiating (3), we get:

$$f'''(y) = 3 \cdot 2 \cdot 1 \cdot A_3 + 4 \cdot 3 \cdot 2 \cdot A_4(y - x) \cdots \tag{6}$$

Plugging in $y = x$ again, we get:

$$f'''(x) = 3 \cdot 2 \cdot 1 \cdot A_3 \tag{7}$$

Therefore, we get:

$$A_3 = f'''(x)/3! \tag{8}$$

Differentiating (6) and plugging in $y = x$ yields:

$$A_4 = f''''(x)/4! \tag{9}$$

We may easily generalize this to A_n for a larger natural number n.

If we plug in the values of A_2,A_3,A_4 which we obtained above into (1), we get:

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2!}(y-x)^2 + \frac{f'''(x)}{3!}(y-x)^3 + \frac{f''''(x)}{4!}(y-x)^4 + \cdots \tag{10}$$

Of course, there is an assumption here. The assumption is that the right-hand side converges. A necessary condition for this to happen is the following. (This is not a sufficient condition, however.)

$$\lim_{n\to\infty} \frac{f^n(x)}{n!}(y-x)^n = 0 \tag{11}$$

where $f^{(n)}(x)$ means that we have differentiated $f(x)$ n times.

In other words, the infinite sum doesn't converge, unless the terms that you are adding approach zero.

When you Taylor expand around 0, you get the following formula.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots \tag{12}$$

To confirm it, you need to plug x for y and 0 for x in the formula (10). This special case of Taylor series is also known as "Maclaurin series."

Here are some lists of Maclaurin series of several functions. Since they are not difficult, I recommend that you confirm them by yourself. Notice that $\mathcal{O}(x^n)$ here refers to the order x^n and higher. This notation is pretty common when we write down remainders in Taylor series.

$$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5! + \mathcal{O}(x^6) \tag{13}$$

(convergent for any number)

$$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6 + \mathcal{O}(x^7) \tag{14}$$

(convergent for $-1 < x \leq 1$)

$$\sin x = x - x^3/3! + x^5/5! - x^7/7! + x^9/9! - x^{11}/11! + \mathcal{O}(x^{13}) \tag{15}$$

(convergent for any number)

To get a better sense of how Taylor series works, see Fig.1. and Fig.2, which are the sums up to 5th terms and 9th terms of Taylor expansions for the sine function, respectively. Compare them with Fig.3 which is the graph of the sine function. You see that as you add more terms it looks more like the sine function. Furthermore, you see that the approximations

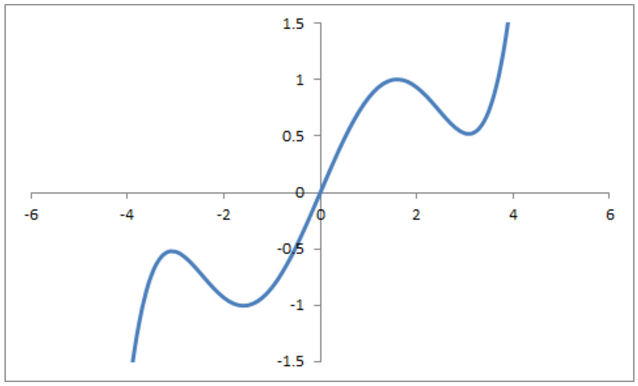

Figure 1: $x - x^3/3! + x^5/5!$

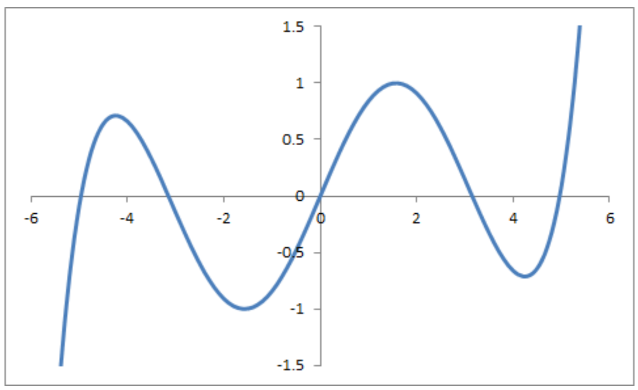

Figure 2: $x - x^3/3! + x^5/5! - x^7/7! + x^9/9!$

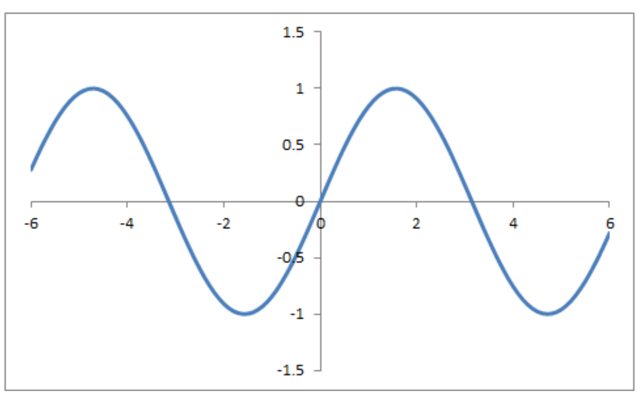

Figure 3: $\sin x$

are best when x is around zero, since we Taylor expanded sine function around $x = 0$ (i.e. we plugged in $x = 0$ in (10))

Problem 1. Derive the Maclaurin series for $\cos x$ up to x^{10} order. (If you could successfully derive the Taylor series for $\sin x$ this shouldn't be hard.)

Problem 2. Let $f(x) = (1+x)^n$. First, calculate

$$f'(x) =?, \qquad f''(x) =?, \qquad f'''(x) =?, \qquad f''''(x) =? \tag{16}$$

Then, by using Maclaurin series, show that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots \tag{17}$$

This is exactly Newton's formula which we explained in "The imagination in mathematics: "Pascal's triangle, combination, and the Taylor series for square root""

Problem 3. In this problem, you are asked to derive (14) alternatively, by using the following two formulas:

$$\ln(1+x) = \int \frac{1}{1+x}dx \tag{18}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \mathcal{O}(x^6) \tag{19}$$

Notice that (19) is not valid when $|x| > 1$, as it doesn't converge in such a case. Same is true for (14) when $|x| > 1$.

A comment. Suppose you encounter divergent series such as the right-hand side of (14) or (19) when $|x| > 1$. What should you do? Should you give up? The way out is converting such series into a form similar to the one of left-hand side of (14) or (19). One of such methods is invented by the French mathematician Émile Borel, and called "Borel summation method." I never knew about such a method until I learned it at a physics winter school long after I got a Master's degree.

Summary

- Taylor series is given by

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2!}(y-x)^2 + \frac{f'''(x)}{3!}(y-x)^3 + \frac{f''''(x)}{4!}(y-x)^4 + \cdots$$

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$
- $\ln(1+x) \approx x$
-

$$\sin x = x - \frac{x^3}{3!} + \mathcal{O}(x^5), \qquad \cos x = 1 - \frac{x^2}{2!} + \mathcal{O}(x^4)$$