Non-Abelian gauge theory in differential forms

In this article, we introduce how non-Abelian gauge theory can be expressed in the language of differential forms often used in mathematics.

Earlier, the covariant derivative was defined as follows:

$$D_\mu \psi = (\partial_\mu - igA_\mu)\psi \tag{1}$$

Here we will re-express the above equation in forms as:

$$D = d + A \tag{2}$$

Here we have suppressed the indices (i.e. μ) and absorbed the factor $-ig$ into the definition of A.

Now $[D_\mu, D_\nu]\psi$ can be expressed as follows:

$$\begin{aligned} D^2\psi &= (d+A)(d+A)\psi = (d+A)(d\psi + A\wedge\psi) \\ &= d^2\psi + A\wedge d\psi + dA\wedge\psi - A\wedge d\psi + A\wedge A\psi \\ &= (dA + A\wedge A)\psi \end{aligned} \tag{3}$$

where from the first line to the second line we have used the fact that A is a one-form and the relation

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg\alpha}(\alpha\wedge d\beta) \tag{4}$$

Summarizing, we can define the field strength F as follows:

$$F = dA + A\wedge A \tag{5}$$

Also, at this point, it might seem odd that $A\wedge A$ in the above equation is not zero, even though the wedge-product of an object is taken with itself. This term can be expressed in component form as follows:

$$\begin{aligned} A\wedge A &= A_\mu dx^\mu \wedge A_\nu dx^\nu \\ &= A_\mu A_\nu dx^\mu\wedge dx^\nu = A_\nu A_\mu dx^\nu \wedge dx^\mu = -A_\nu A_\mu dx^\mu\wedge dx^\nu \\ &= \frac{A\wedge A}{2} + \frac{A\wedge A}{2} = \frac{A_\mu A_\nu}{2} - \frac{A_\nu A_\mu}{2} dx^\mu dx^\nu \\ &= \frac{1}{2}[A_\mu, A_\nu] dx^\mu\wedge dx^\nu \end{aligned} \tag{6}$$

Using this, we can also express (5) as follows:

$$F = dA + \frac{1}{2}[A, A] \tag{7}$$

We can also derive the equation of motion that F satisfies

$$
\begin{aligned}
dF &= d^2A + dA \wedge A - A \wedge dA = dA \wedge A - A \wedge dA \\
A \wedge F &= A \wedge dA + A \wedge A \wedge A \\
F \wedge A &= dA \wedge A + A \wedge A \wedge A
\end{aligned} \tag{8}
$$

Therefore we conclude:

$$
\begin{aligned}
dF + A \wedge F - F \wedge A &= 0 \\
dF + [A, F] &= 0
\end{aligned} \tag{9}
$$

This is called the Bianchi identity. The last step can be shown as follows:

$$
A \wedge F - F \wedge A = A_\lambda(\frac{1}{2}F_{\mu\nu})dx^\lambda \wedge dx^\mu \wedge dx^\nu - (\frac{1}{2}F_{\mu\nu})A_\lambda dx^\mu \wedge dx^\nu \wedge dx^\lambda \tag{10}
$$

$$
= [A_\lambda, \frac{1}{2}F_{\mu\nu}]dx^\lambda \wedge dx^\mu \wedge dx^\nu = [A, F] \tag{11}
$$

Also, if we obtain the equation of motion by varying the Yang-Mills action (i.e. $\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$), we obtain:

$$
d * F + [A, *F] = 0 \tag{12}
$$

We will not show the proof. In the presence of charge, as we considered in our earlier article "The Maxwell Lagrangian and its equation of motion," the right-hand side of (12) (i.e. 0) is simply replaced by J.

Finally, in the earlier article "Non-Abelian gauge theory," we mentioned the following infinitesimal gauge transformation:

$$
A^a_\mu \to A^a_\mu + \partial_\mu\theta^a - gf^{abc}\theta^b A^c_\mu \tag{13}
$$

$$
F^a_{\mu\nu} \to F^a_{\mu\nu} - gf^{abc}\theta^b F^c_{\mu\nu} \tag{14}
$$

In the differential forms language these are translated into:

$$
A \to A + d\theta + [A, \theta] \tag{15}
$$

$$
F \to F + [F, \theta] \tag{16}
$$

These equations can be shown by using following facts:

$$
A = -igA_\mu dx^\mu = -igA^a_\mu T^a dx^\mu \tag{17}
$$

$$
\theta = -ig\theta^b T^b \tag{18}
$$

$$
F = (-ig)\frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = (-ig)\frac{1}{2}F^a_{\mu\nu}T^a dx^\mu \wedge dx^\nu \tag{19}
$$

$$
[T^a, T^b] = if^{abc}T^c \tag{20}
$$

where we have revived the $-ig$ factor that we had absorbed in the definition of A in (2).

Notice that (15) is exactly $\omega^a \to \omega^a + D\Lambda^a$ obtained in "Gauge transformation in dreibein."

Problem 1. Convince yourself that (12) is equivalent to $D_\mu F^{\mu\nu} = 0$ obtained in our earlier article.

Summary

- In differential form notation, a covariant derivative is often written as

$$D = d + A$$

- Then, the field strength is given by

$$D^2 = F = dA + A \wedge A$$

- The Bianchi identity is given by

$$dF + A \wedge F - F \wedge A = dF + [A, F] = 0$$

- The equation of motion is given by

$$d * F + [A, *F] = 0$$

- Under the gauge transformation, we have

$$A \to A + D\theta$$

$$F \to F + [F, \theta]$$