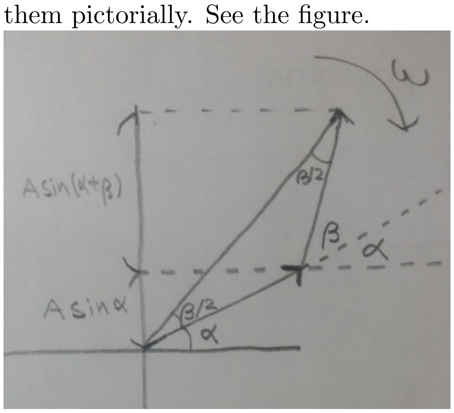
Young's interference experiment, revisited

In our earlier article on Young's interference experiment, we obtained, at which location on the screen the constructive interference and the destructive interference would occur respectively. We also found, the amplitude is doubled at the location where the constructive interference occurs, and the amplitude is zero at the location where the destructive interference occurs. However, what we did not consider in that article was the case somewhere between; we didn't find the amplitude at the location where the interference is neither totally constructive nor totally destructive. In this article, we will find the amplitude for such locations and by stepping further, the intensity.

In Young's interference experiment, we have two slits. The displacement of light wave at a certain location on the screen is the sum of the displacement of light wave emerging from the first slit and the one emerging from the second slit. If the distance from a certain location on the screen to the first slit is s_1 and the one to the second slit s_2, the displacement of light wave due to the first slit can be represented as $A\sin(ks_1 - \omega t)$ while the one due to the second slit can be represented as $A\sin(ks_2 - \omega t)$. The resulting displacement is sum of them. OK. Let's add them. Before adding them, let's relabel the variables for convenience. Let's say $ks_1 - \omega t = \alpha$, and $k(s_2 - s_1) = \beta$. Then, the displacement E is given by

$$E = A\sin\alpha + A\sin(\alpha + \beta) \tag{1}$$

We could look up trigonometric identities and do the sum, but it's more interesting if we add them pictorially. See the figure.



We are adding two vectors with magnitude A. The vectors are denoted by arrows. Then, the y component of the sum of these two vectors is exactly what we want. Notice also that the addition diagram of these two vectors form an isosceles triangle, as their two sides are same, being A. We also know that the two angles of an isosceles triangle are same, and the sum of all angles of a triangle is 180 degrees (or π radian). Therefore, each of the same two angles must be given by $\beta/2$. Therefore, the other side of the isosceles triangle is given by $2A\cos(\beta/2)$. Therefore, we conclude

$$E = 2A\cos(\beta/2)\sin(\alpha + \beta/2) \tag{2}$$

Remember also that α, $\alpha + \beta/2$ and $\alpha + \beta$ are constantly decreasing with rate ω per second because α is given by $ks_1 - \omega t$. This means that the whole diagram is constantly rotating clockwise with the rate ω radian per second.

Using $s_2 - s_1 = d\sin\theta$ and $k = 2\pi/\lambda$ (see the earlier article for the definition of d, θ, λ), we obtain

$$E = 2A\cos\left(\frac{d}{\lambda}\sin\theta\right)\sin(ks_1 + \frac{d}{\lambda}\sin\theta - \omega t) \tag{3}$$

Here, we see that $\sin(ks_1 + d\sin\theta/\lambda - \omega t)$ is the oscillating part, as ωt is in it, and t is constantly increasing. Therefore, the amplitude is given by $2A\cos(d\sin\theta/\lambda)$. In other words, the magnitude of the addition of the two vectors is the amplitude.

So far, we have only talked about amplitude. Now, let's talk about the intensity. It is a general rule that the intensity, defined by the energy transported during unit time, is proportional to the square of the amplitude. In our case, the displacement of light wave is given by the value of electric field or magnetic field (which is proportional to the electric field anyway), and it can be shown that energy transported is proportional to the square of the electric field. Notice also that intensity is never negative while displacement can be negative, as something squared is never negative.

In our case, if the intensity of light due to the single slit is I_0, and the intensity of light due to the double slit is I, the proportionality condition can be written as

$$\frac{I}{I_0} = \left(\frac{2A\cos\left(\frac{d}{\lambda}\sin\theta\right)}{A}\right)^2 \tag{4}$$

which implies

$$I = 4I_0\cos^2\left(\frac{d}{\lambda}\sin\theta\right) \tag{5}$$

At first glance, this formula may look surprising. For constructive interference, the intensity is $4I_0$, while we get only $2I_0$ if we sum the intensity of two sources, each with I_0. It may seem energy is not conserved, but created. However, notice that for destructive interference the intensity is 0. So, we can easily see that the energy has been re-distributed; the energy that would have gone to the location where the destructive interference occurs if there were no interference at all, has gone to where the constructive interference occurs. We can actually see this more precisely. As we have

$$\cos^2(\beta/2) = \frac{1+\cos\beta}{2} \tag{6}$$

the average of $\cos^2(\beta/2)$ is half, as the average of $\cos\beta$ is zero. Therefore, the average of I is $4I_0$ times half, which is $2I_0$. This makes sense.

Problem 1. Draw the corresponding diagrams for totally constructive interference and for totally destructive interference for double slit experiment (i.e. Young's interference experiment)

Problem 2. Consider a triple slit experiment. The distance between two neighboring slit is d each, and let's say we shoot light with wavelength λ. If θ is defined as in our earlier article, at what θ would we have totally destructive interference? Solve this problem by drawing a diagram. (Hint[1])

Summary

- A superposition of waves, such as in Young's interference experiment, can be quantitatively analyzed by summing each displacement of wave contributing to the superposition.
- Intensity is proportional to the square of the amplitude.

[1]The sum of the three vectors must be zero, which means that the diagram composed of the three vectors is an equilateral triangle.