Addition, subtraction for differentiation and Leibniz's rule

Let a function $h(x)$ given by a sum of two functions $f(x)$ and $g(x)$. In other words, $h(x) = f(x) + g(x)$. Then, it is easy to see that

$$\begin{aligned}
\frac{dh}{dx} &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \frac{df}{dx} + \frac{dg}{dx} \qquad (1)
\end{aligned}$$

Therefore, the derivative of sum of two functions is equal to sum of derivative of each function.

Convince yourself the followings as well. If $k(x) = f(x) - g(x)$,

$$\frac{dk}{dx} = \frac{df}{dx} - \frac{dg}{dx} \qquad (2)$$

If $m(x) = cf(x)$, where c is a constant, we have:

$$\frac{dm}{dx} = c\frac{df}{dx} \qquad (3)$$

Now, what would be the derivatives of functions multiplied together? Let $p(x) = f(x)g(x)$ and let's find dp/dx.

$$\begin{aligned}
\frac{dp(x)}{dx} &= \lim_{\Delta x \to 0} \frac{p(x+\Delta x) - p(x)}{\Delta x} \\
\frac{d(f(x)g(x))}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} g(x+\Delta x) + \lim_{\Delta x \to 0} f(x) \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{df(x)}{dx} g(x+\Delta x) + f(x)\frac{dg(x)}{dx} \\
\frac{d(f(x)g(x))}{dx} &= \frac{df(x)}{dx} g(x) + f(x)\frac{dg(x)}{dx} \qquad (4)
\end{aligned}$$

This rule is called "Leibniz's rule."

Deriving Leibniz's rule in such a cumbersome way may not quite illuminate what is actually happening. So, let's derive it in another way, a way your math professor in university would not allow. When x becomes $x + dx$, f becomes $f + df$, g becomes $g + dg$ and p becomes $p + dp$. In other words,

$$p = fg \qquad (5)$$

$$p + dp = (f + df)(g + dg) = fg + df\,g + f\,dg + df\,dg \tag{6}$$

If we plug (5) into the above equation, we get

$$\begin{aligned} p + dp &= p + df\,g + f\,dg + df\,dg \qquad (7) \\ dp &= df\,g + f\,dg + df\,dg \qquad (8) \end{aligned}$$

If we divide the above formula by dx, we get

$$\frac{dp}{dx} = \frac{df}{dx}g + f\frac{dg}{dx} + \frac{df}{dx}dg \tag{9}$$

However, we know that dg is infinitesimal, so the last term is zero. Thus, we get

$$\frac{dp}{dx} = \frac{df}{dx}g + f\frac{dg}{dx} \tag{10}$$

which is exactly (4).

Problem 1. Show the following. (Hint[1])

$$\frac{d(f(x)g(x)h(x))}{dx} = \frac{df(x)}{dx}g(x)h(x) + f(x)\frac{dg(x)}{dx}h(x) + f(x)g(x)\frac{dh(x)}{dx}$$

Summary

- $\dfrac{d(f+g)}{dx} = \dfrac{df}{dx} + \dfrac{dg}{dx}$
- $\dfrac{d(f-g)}{dx} = \dfrac{df}{dx} - \dfrac{dg}{dx}$
- If $m(x) = cf(x)$ where c is a constant,

$$\frac{dm}{dx} = c\frac{df}{dx}$$

- Leibniz's rule is given by

$$\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}$$

[1] Let $g(x)h(x) = u(x)$ and calculate $d(f(x)u(x))/dx$. You will need to use Leibniz's rule twice.