Addition and subtraction rules for trigonometric functions

Having introduced the rotation in Cartesian coordinate system, we are on a concrete position to derive addition and subtraction rule for trigonometric functions.

See Fig. 1. We rotate a point given by $(r\cos\alpha, r\sin\alpha)$ by the angle θ. Then the coordinate for the rotated point will be given by $(r\cos(\alpha+\theta), r\sin(\alpha+\theta))$. Plugging $x = r\cos\theta$ and $y = r\sin\theta$, $x' = r\cos(\alpha+\theta)$ and $y' = r\sin(\alpha+\theta)$ in the rotation formula we just obtained in the previous article, we get the following:

$$r\cos(\alpha+\theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \tag{1}$$

$$r\sin(\alpha+\theta) = r\cos\alpha\sin\theta + r\sin\alpha\cos\theta \tag{2}$$

Dividing the both-hand sides by r, and rearranging a little bit, we get the following called "addition rules" for trigonometric functions.

$$\sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta \tag{3}$$

$$\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta \tag{4}$$

The subtraction rules are also easily obtained by substituting θ for $-\theta$ in the addition rules, and using the fact that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Explicitly, we get:

$$\sin(\alpha+(-\theta)) = \sin\alpha\cos(-\theta) + \cos\alpha\sin(-\theta) = \sin\alpha\cos\theta - \cos\alpha\sin\theta$$

$$\cos(\alpha+(-\theta)) = \cos\alpha\cos(-\theta) - \sin\alpha\sin(-\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$$

This completes the proof.

Problem 1. Obtain the values for $\sin 75°$ and $\sin 15°$. (Hint[1])

Problem 2. Prove $\sin(2\theta) = 2\sin\theta\cos\theta$.

Problem 3. Prove $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

Problem 4. Prove $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.

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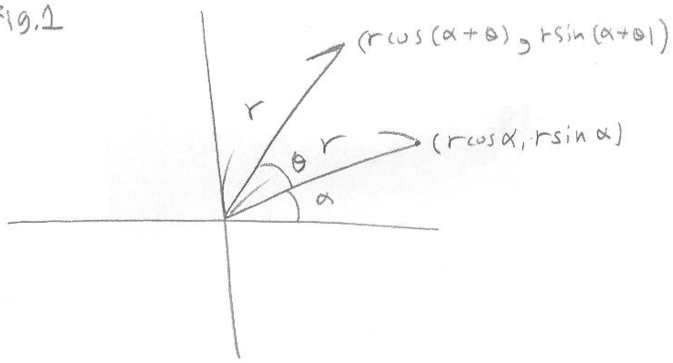
[1] $\sin 75° = \sin(45° + 30°)$, $\quad \sin 15° = \sin(45° - 30°)$

Fig.1



Summary

- The cosines and sines of the sum of two angles are expressible in terms of the cosines and sines of the original two angles.