Algebraic topology

Having learned algebra and topology, we can use algebra to analyze topology. See Fig. 1. Consider loops that start at the red point and end at the red point, assuming that it is forbidden that they enter the hole marked in blue. In the figure, there are three such loops. The black loop, the gray loop, and the brown loop. The black loop doesn't encircle the blue hole, the gray loop encircles the blue hole once, and the brown loop encircles the blue hole twice. Now, think about whether it is possible to smoothly deform the black loop to change it into the gray loop. It is impossible, unless you cut somewhere in the black loop, wind it through the blue hole, and connect it again. Then, of course, it is no longer smooth deformation. The same can be said about the brown loop. It is impossible to change the black loop or the gray loop into the brown loop by a smooth deformation. In other words, you see that the three loops are topologically different. They cannot change from one to the other. Thus, a classification of such loops gives us information about the topology of space. In the case of Fig. 1, the space is a space with one hole, which is denoted in blue.

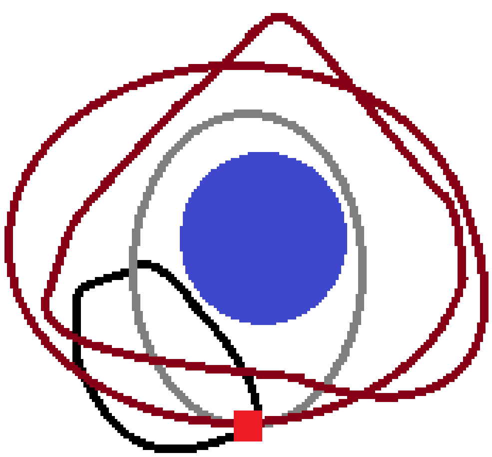

Figure 1: The gray, black, and brown loops are topologically distinct.

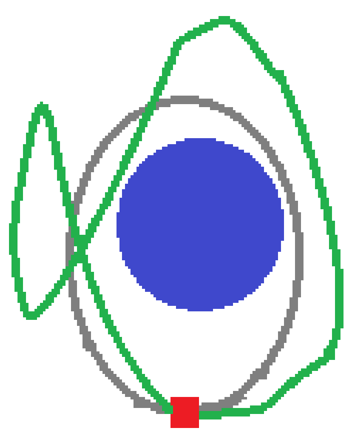

Figure 2: The gray loop and the green loop are topologically the same.

Now, let's use the concept of group to systematically talk about the loops that start at the red point and end at the red point. First of all, two loops are counted as the same loop, if one can be smoothly deformed into the other. You can regard the loop as a string or a thread. If you can change one loop into the other without cutting off the string or the thread, they are counted as the same loop. For example, the green loop and the gray loop in Fig. 2

are counted as the same loop.

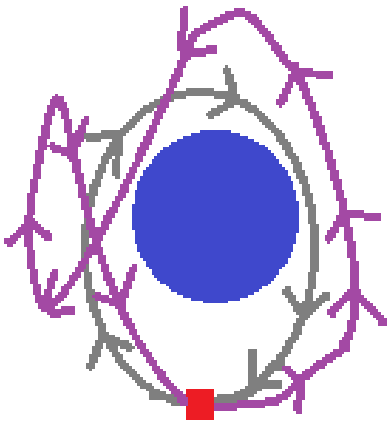

Figure 3: giving orientions to loops

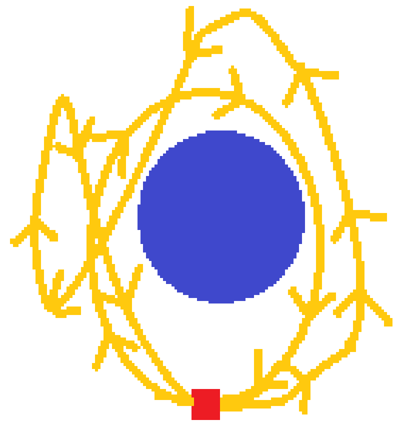

Figure 4: gray• purple=orange

Second, we need to define group operation. To this end, we first need to give orientation to the loop. You will later see why this step is necessary. See Fig. 3. We gave orientation to each loop. Now, you see that the purple loop has the opposite orientation of the gray loop. In this sense, the purple loop is counted as *different* from the gray loop. You may be able to change the gray loop into the purple loop, if you don't care about the orientation, but it's impossible to change the gray loop into the purple loop, if you care about the orientation. This is because, the gray loop winds the blue hole in clockwise direction, while the purple loop winds the blue hole in anti-clockwise direction. Given this, we can now define group operation. It is given by joining two loops. For example let's join the gray loop and the purple loop in Fig. 3. You start from the gray loop along the arrow of the orientation, you arrive at the red point, then you continue to follow the purple loop along the arrow of the orientation. Then, finally you come back to the red point again. See Fig. 4. The result is the orange loop. It is perfectly a valid loop, because it starts at the red point and ends at the red point. Of course, it also passes the red point in the middle of loop, which does not matter at all, but for clarity, we can smoothly move this middle point, if we want to. See the orange loop in Fig. 5, which is the same loop as the orange loop in Fig. 4. In other words,

$$\text{gray} \bullet \text{purple} = \text{orange} \tag{1}$$

Now, let's consider the loop that starts with purple, and ends with gray. That is the pink loop in Fig. 6. In other words, we have

$$\text{purple} \bullet \text{gray} = \text{pink} \tag{2}$$

So, we defined the group operation. Then, what is the identity element? That is the loop that can be shrunken to the red point like the black loop in Fig. 1. If you add black loop,

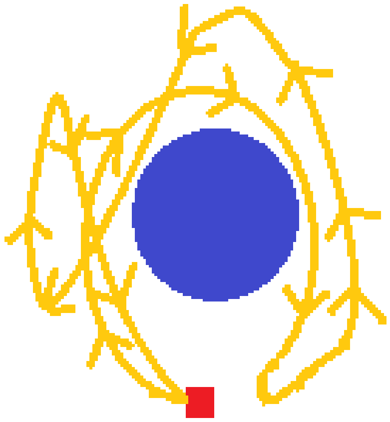

Figure 5: the same loop as Fig. 4

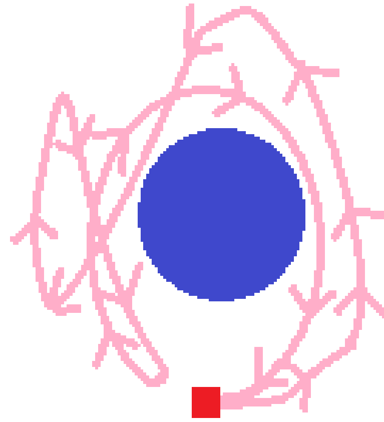

Figure 6: purple• gray=pink

either before or after the original loop, you will get the topologically same loop, because the black loop, wrapping no blue hole, can be shrunken to the red point. It is just as if the black loop does nothing and can be ignored.

Then, what is the inverse element? It is the same loop with orientation inverted. For example, we have seen that the gray loop wraps the blue hole in clock-wise direction. The inverse of the gray loop would be the same loop, but wrapping the blue hole anti-clock-wise, as the arrow (i.e., the orientation) is reverted. Then, it is very easy to check that you get the identity element, the black loop, or a point on the red point, if you combine a loop with the same loop with the opposite orientation. In a nutshell, you wind the blue hole once, then unwind it, which results in the identity element. Actually, it is very easy to check that the orange loop in Fig. 5 and the pink loop in Fig. 6 is an identity element. You can smoothly deform them to the black loop in Fig. 1. In other words, in Fig. 3, we have

$$\text{gray}^{-1} = \text{purple} \tag{3}$$

Similarly, one can easily check the associativity of this group action. If you connect three loops, it doesn't matter which two loops you connect first, as long as you don't change the order of the loop.

What we have dealt in this article is the "fundamental group." We have played with the fundamental group of space that has one hole, which we denoted in blue. So, what is the fundamental group of this space? We can classify each loop by one integer. For example, the identity element corresponds to zero, because it doesn't wrap the blue hole. The gray loop which wraps the blue hole once in clockwise direction corresponds to 1. The purple loop which wraps the blue hole once in counter clockwise direction corresponds to -1. In this way, we can assign a unique integer to each loop. If it wraps n times in clockwise direction, we assign n. If it wraps n times in anti-clockwise direction, we assign $-n$. This number is often called "winding number," which plays an important role in string theory.

Now, it's easy. The group action adds the winding number. For example,

$$\text{gray} \bullet \text{purple} = \text{orange}, \quad \text{corresponds to} \quad 1 + (-1) = 0 \tag{4}$$

$$\text{purple} \bullet \text{gray} = \text{pink}, \quad \text{corresponds to} \quad (-1) + 1 = 0 \tag{5}$$

Similarly, if you connect a loop that wraps the blue hole clockwise 3 times with another loop that wraps the blue hole anticlockwise 2 times, you will get a loop that wraps the blue hole clockwise 1 time.

It is also clear why we introduced the orientation. If we didn't, we would not be able to find the inverse element, which is necessary for the group axiom. In a later article, we will introduce other groups that deal with the topology of space. They are homology and de Rham cohomology.

You usually learn algebraic topology only when you study math in graduate school. I would be sorry, if you missed this wonderful topic because you didn't plan to study math in graduate school, so I wrote this article.

Problem 1. Is the fundamental group of a plane with one hole removed, which we considered in this article, Abelian?

Problem 2. Explain why the fundamental group of $\mathbb{R}^2$ (i.e., 2-dimensional plane with no hole) only consists of the identity element.

Problem 3. Explain why the fundamental group of 2-sphere only consists of the identity element.

Problem 4. (Very challenging) Is the fundamental group of plane with two holes Abelian?

Summary

- The fundamental group of a plane with one hole removed is given by integers.