## Re-visiting angular momentum conservation in central force

As promised in our earlier articles, we will re-visit angular momentum mathematically in this article, and re-derive angular momentum conservation in central force.

In particular, we will upgrade the angular momentum to a vector. Remember that the magnitude of angular momentum depends on how fast you rotate. So, it is natural that the direction of the angular momentum must depend on which direction you rotate; if you move on $x-y$ plane (i.e. you rotate around $z$ axis), your angular momentum is in $z$ direction, and if you move on $y-z$ plane (i.e. you rotate around $x$ axis), your angular momentum is in $x$ direction, if you move on $z-x$ plane (i.e. you rotate around $y$ axis), your angular momentum is in $x$-direction.

Now, let me state how the angular momentum is defined mathematically. Angular momentum $\vec{L}$ is defined as follows by using cross product:

$$
\begin{equation*}
\vec{L}=\vec{r} \times \vec{p} \tag{1}
\end{equation*}
$$

where $\vec{r}$ is the position vector and $\vec{p}$ the momentum vector. If we express the position vector and the momentum vector in Cartesian coordinate as follows,

$$
\begin{gather*}
\vec{r}=x \hat{i}+y \hat{j}+z \hat{k}  \tag{2}\\
\vec{p}=p_{x} \hat{i}+p_{y} \hat{j}+p_{z} \hat{k} \tag{3}
\end{gather*}
$$

and call the $x, y, z$-components of angular momentum by $L_{x}, L_{y}, L_{z}$ as follows

$$
\begin{equation*}
\vec{L}=L_{x} \hat{i}+L_{y} \hat{j}+L_{z} \hat{k} \tag{4}
\end{equation*}
$$

we have (Problem 1. Check this!)

$$
\begin{align*}
& L_{x}=y p_{z}-z p_{y}  \tag{5}\\
& L_{y}=z p_{x}-x p_{z}  \tag{6}\\
& L_{z}=x p_{y}-y p_{x} \tag{7}
\end{align*}
$$

We can also check that the direction mentioned earlier coincides. For example, if you move on $x-y$ plane, we have $z=p_{z}=0$ which implies $L_{x}=L_{y}=0$ and $L_{z}$ is the only non-zero component of angular momentum; the angular momentum is in $z$-direction.

Now, let's consider such a case but using polar coordinate.


Figure 1: Angular momentum

$$
\begin{gather*}
\vec{r}=r \cos \theta \hat{i}+r \sin \theta \hat{j}  \tag{8}\\
\vec{p}=m \vec{v}=m v_{x} \hat{i}+m v_{y} \hat{j} \tag{9}
\end{gather*}
$$

where

$$
\begin{align*}
& v_{x}=\frac{d x}{d t}=\frac{d r}{d t} \cos \theta-r \frac{d \theta}{d t} \sin \theta  \tag{10}\\
& v_{y}=\frac{d y}{d t}=\frac{d r}{d t} \sin \theta+r \frac{d \theta}{d t} \cos \theta \tag{11}
\end{align*}
$$

Plugging this into (7), we obtain (Problem 2. Check this!):

$$
\begin{equation*}
\vec{L}=m r^{2} \dot{\theta} \hat{k} \tag{12}
\end{equation*}
$$

So, this coincides with our earlier formula! Also, notice that $\vec{L}$ is in $z$-direction if you move counter-clockwise (i.e. $\dot{\theta}>0$ ) and is in negative $z$-direction if you move clockwise (i.e. $\dot{\theta}<0$ ).

Actually, there is an easier way to obtain the above formula. Recall the definition of cross product:

$$
\begin{equation*}
|\vec{L}|=|\vec{r} \| \vec{p}| \sin \theta \tag{13}
\end{equation*}
$$

where $\theta$ is the angle between $\vec{r}$ and $\vec{p}$. See Fig.l. $\vec{p}$ is composed of two parts. The one parallel to $\vec{r}$ (denoted here as $|\vec{p}| \cos \theta)$ and the one orthogonal to $\vec{r}$ (denoted here as $|\vec{p}| \sin \theta)$. The one parallel to $\vec{r}$ doesn't contribute to the cross product, while only the orthogonal one does. However, the orthogonal velocity is $r \dot{\theta}$. Therefore, $|\vec{L}|=m r^{2} \dot{\theta}$. It also goes without saying that the direction of the angular momentum is $\hat{k}$ direction since both position and momentum vectors lie on $x-y$ plane.

Let me explain once more using polar coordinate. We have

$$
\begin{equation*}
\vec{r}=r \hat{r}, \quad \vec{p}=m \vec{v}=m \dot{r} \hat{r}+m r \dot{\theta} \hat{\theta} \tag{14}
\end{equation*}
$$

Then, $\vec{L}=\vec{r} \times \vec{p}=m r^{2} \dot{\theta} \hat{k}$, if we use $\hat{r} \times \hat{r}=0$, and $\hat{r} \times \hat{\theta}=\hat{k}$.
So, why all this fuss about angular momentum? It is because in central force (i.e. the force is in the $\hat{r}$ direction, toward center) the angular momentum is conserved. Let's prove this:

$$
\begin{align*}
\frac{d \vec{L}}{d t} & =\frac{d \vec{r}}{d t} \times \vec{p}+\vec{r} \times \frac{d \vec{p}}{d t}  \tag{15}\\
& =\frac{d \vec{r}}{d t} \times m \frac{d \vec{r}}{d t}+\vec{r} \times \vec{F}  \tag{16}\\
& =\vec{r} \times \vec{F} \tag{17}
\end{align*}
$$

where in the first step we used Leibniz rule and in the second step we use the fact that a vector cross producted itself is zero. (Those of you who know torque will notice that the above formula is exactly torque, since $\tau=r F \sin \theta$.) However, notice also that $\vec{r} \times \vec{F}$ should be zero as well since both $\vec{r}$ and $\vec{F}$ are in radial direction. So, angular momentum is indeed conserved! In other words,

$$
\begin{equation*}
m r^{2} \dot{\theta}=\text { constant } \tag{18}
\end{equation*}
$$

Remember also that angular momentum is a vector, so it has both magnitude and direction. Therefore, if angular momentum is conserved, not only its magnitude but also its direction is conserved. This has a far reaching consequence. If the direction of the angular momentum is along say, $z$-axis, it means that the orbit of the planet is in $x-y$ plane. As long as the direction of angular momentum remains along $z$-axis, the orbit of the planet will remain in $x-y$ plane. More generally, as long as the angular momentum is conserved, the orbit of the planet will remain in the same plane.

Finally, we will consider the conservation of total angular momentum. Consider the solar system, in which the Sun rotates around itself, and planets orbit around the Sun. Now, consider planet $A$. It will receive force from the Sun, and this will not change the angular momentum of planet $A$, as we just argued. However, there are additional gravitational forces from the other planets. They do change the angular momentum of planet $A$ as they exert torque on the planet $A$. However, as we will show now, the total angular momentum, i.e. the sum of the angular momentum of all the planets and the Sun will remain constant, if there is no external force acting on the solar system.

Let's denote the mass, the position and the velocity of each object (i.e. the planets and the Sun) by $m_{i}, \vec{r}_{i}$, and $\vec{v}_{i}$. Then, the total angular momentum is given by

$$
\begin{equation*}
\vec{L}_{\text {total }}=\sum_{i} \vec{r}_{i} \times m_{i} \vec{v}_{i} \tag{19}
\end{equation*}
$$

Taking the time derivative, we get

$$
\begin{equation*}
\frac{d \vec{L}_{\text {total }}}{d t}=\sum_{i} \vec{v}_{i} \times m_{i} \vec{v}_{i}+\sum_{i} \vec{r}_{i} \times \vec{F}_{i}=\sum_{i} \vec{r}_{i} \times \vec{F}_{i} \tag{20}
\end{equation*}
$$

where $\vec{F}_{i}$ is the force exerted on object $i$. Let's denote the force exerted on object $i$ by object $j \vec{F}_{i j}$. Then, we have

$$
\begin{equation*}
\vec{F}_{i}=\sum_{j} F_{i j} \tag{21}
\end{equation*}
$$

Using this relation, (20) can be re-expressed as

$$
\begin{equation*}
\frac{d \vec{L}_{\text {total }}}{d t}=\sum_{i} \sum_{j} \vec{r}_{i} \times \vec{F}_{i j} \tag{22}
\end{equation*}
$$

Renaming the label $i$ by $j$ and the label $j$ by $i$, we get ${ }^{1}$

$$
\begin{equation*}
\frac{d \vec{L}_{\text {total }}}{d t}=\sum_{j} \sum_{i} \vec{r}_{j} \times \vec{F}_{j i} \tag{23}
\end{equation*}
$$

Now, remember Newton's third law it says $\vec{F}_{j i}=-\vec{F}_{i j}$. Using this relation along with $\sum_{j} \sum_{i}=\sum_{i} \sum_{j}$, we get

$$
\begin{equation*}
\frac{d \vec{L}_{\mathrm{total}}}{d t}=\sum_{i} \sum_{j}-\vec{r}_{j} \times \vec{F}_{i j} \tag{24}
\end{equation*}
$$

Adding (22) and (24), we get

$$
\begin{equation*}
2 \frac{d \vec{L}_{\text {total }}}{d t}=\sum_{i} \sum_{j}\left(\vec{r}_{i}-\vec{r}_{j}\right) \times \vec{F}_{i j} \tag{25}
\end{equation*}
$$

Notice that $\vec{r}_{i}-\vec{r}_{j}$ is a vector pointing from object $i$ to object $j$. As $\vec{F}_{i j}$ is directed along this direction (i.e. parallel to $\left.\vec{r}_{i}-\vec{r}_{j}\right)$, we obtain $\left(\vec{r}_{i}-\vec{r}_{j}\right) \times \vec{F}_{i j}$ is zero. Thus, we conclude

$$
\begin{equation*}
\frac{d \vec{L}_{\text {total }}}{d t}=0 \tag{26}
\end{equation*}
$$

Notice that the treatment here is general enough; all we assumed was that the force between two objects is along the line that connects them. This is true not just for gravitational force, but for most forces.

Problem 3. Why is the most angular momentum of the solar system due to the planets, not due to the Sun, even though the Sun is much heavier than the planets?

Problem 4. Consider the case in which there are external forces acting on objects in the system in addition to the internal forces between the object in the system. In other words,

$$
\begin{equation*}
\vec{F}_{i}=\vec{F}_{i(\mathrm{ext})}+\sum_{j} \vec{F}_{i j} \tag{27}
\end{equation*}
$$

where $\vec{F}_{i(\text { ext })}$ denotes the force exerted on object $i$ by another object outside of the system. Then, show that the time derivative of the total angular momentum of the system is given by

$$
\begin{equation*}
\frac{d \vec{L}_{\mathrm{total}}}{d t}=\sum_{i} \vec{r}_{i} \times \vec{F}_{i(\mathrm{ext})} \tag{28}
\end{equation*}
$$

In other words, it is given by the external torque.

[^0]
## Summary

- The angular momentum $\vec{L}$ is defined by

$$
\vec{L}=\vec{r} \times \vec{p}
$$

where $\vec{r}$ is the position vector, and $\vec{p}$ is the momentum vector.

- If an object moves in $x-y$ plane, its angular momentum is in $z$-direction.
- As

$$
\frac{d \vec{L}}{d t}=\vec{r} \times \vec{F}
$$

angular momentum is conserved if $\vec{F}$ is in radial direction, i.e., parallel to $\vec{r}$.


[^0]:    ${ }^{1}$ You are allowed to change the labels. If you are not sure why, read our later article "Einstein summation convention."

