Angular momentum addition

Consider two particles with each spin 1/2. What would be the total angular momentum of these two particles? In other words, if the angular momentum of each particle is given by $\vec{L}^{(1)}$ and $\vec{L}^{(2)}$, and if we call $\vec{L} = \vec{L}^{(1)} + \vec{L}^{(2)}$, what would be the eigenvector of L^2 and L_z. This is the question we will answer in this article.

First, for notational convenience we will denote the eigenvectors of zth component of angular momentum of each particle by following notation:

$$|1/2,\ 1/2\rangle = |\uparrow\rangle, \qquad |1/2,\ -1/2\rangle = |\downarrow\rangle \tag{1}$$

For example, using this notation, we can call the first particle with $|\uparrow\rangle$ state and the second particle with $|\downarrow\rangle$ state as follows:

$$|\uparrow\downarrow\rangle \tag{2}$$

One can also easily see that there are total four states as follows:

$$|\uparrow\uparrow\rangle, \qquad |\uparrow\downarrow\rangle, \qquad |\downarrow\uparrow\rangle, \qquad |\downarrow\downarrow\rangle \tag{3}$$

Each of this state is the eigenvector of $L_z = L_z^{(1)} + L_z^{(2)}$. For example,

$$L_z|\uparrow\downarrow\rangle = (L_z^{(1)} + L_z^{(2)})|\uparrow\downarrow\rangle = \frac{\hbar}{2}|\uparrow\downarrow\rangle - \frac{\hbar}{2}|\uparrow\downarrow\rangle = 0 \tag{4}$$

where you can see that $L_z^{(1)}$ acts only on the first particle (i.e. the one with spin up) and $L_z^{(2)}$ acts only on the second particle (i.e. the one with spin down). This relation is obvious since the zth component of the total angular momentum is the sum of zth component of the angular momentum of each particle. Similarly, we can see:

$$\begin{aligned}
L_z|\uparrow\uparrow\rangle &= \hbar|\uparrow\uparrow\rangle \\
L_z|\uparrow\downarrow\rangle &= 0 \\
L_z|\downarrow\uparrow\rangle &= 0 \\
L_z|\downarrow\downarrow\rangle &= -\hbar|\downarrow\downarrow\rangle
\end{aligned}$$

As $j = 1$ (i.e. total angular momentum) has $m = -1,\ 0,\ 1$ (i.e. the zth component of total momentum)it may seem that we have $j = 1$ at first glance. However, this is not correct because in our example the space for $m = 0$ is two-dimensional rather than one-dimensional as it should be. Let's check what happened. To this end, let's first check $|\uparrow\uparrow\rangle = |1,\ 1\rangle$. As we already have its zth momentum is $\hbar$ we only need to check L^2. We have:

$$L_+|\uparrow\uparrow\rangle = L_+^{(1)}|\uparrow\uparrow\rangle + L_+^{(2)}|\uparrow\uparrow\rangle = 0 + 0 = 0 \tag{5}$$

Therefore,

$$L^2|\uparrow\uparrow\rangle = (L_z^2 + \hbar L_z + L_-L_+)|\uparrow\uparrow\rangle = (\hbar^2 + \hbar\hbar)|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle \tag{6}$$

So, we indeed see $j = 1$. Given this, notice

$$L_-|\uparrow\uparrow\rangle = L_-|1,\ 1\rangle = \sqrt{2}\hbar|1,\ 0\rangle \tag{7}$$

On the other hand, we also have:

$$L_-|\uparrow\uparrow\rangle = (L_-^{(1)} + L_-^{(2)})|\uparrow\uparrow\rangle = \hbar|\downarrow\uparrow\rangle + \hbar|\uparrow\downarrow\rangle \tag{8}$$

where you see that $L_-^{(1)}$ acts only on the first particle and remains the second particle intact. And vice versa for $L_-^{(2)}$. Equating the right-hand sides of (7) and (8), we conclude:

$$|1,\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{9}$$

where you see that the factor $1/\sqrt{2}$ ensures that it is properly normalized. (i.e. $\langle 1,\ 0|1,\ 0\rangle = 1$)

Given this, what is the orthogonal state to $|1,\ 0\rangle$, which also has $L_z = 0$? It is given by following:

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{10}$$

Indeed, taking a scalar product this one with (9) yields:

$$\frac{1}{2}(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{1}{2}(1 - 0 + 0 - 1) = 0 \tag{11}$$

Now, we know that (10) is orthogonal to (9), and its z component of angular momentum is zero. What is its j? Taking the same step as showing that $|\uparrow\uparrow\rangle$ has $j = 1$, it is easy to show that in case of (10) we have $j = 0$. (**Problem 1.** Show this)

In conclusion, if you have two particles each of spin with 1/2, the total angular momentum can be $j = 1$ or $j = 0$. It makes sense since spin 1/2 is two dimensional so the total vector space is two times two which is four, and $j = 1$ is three dimensional and $j = 0$ is one dimensional, which again makes four.

The procedure given in this article can be repeated for a general case. Let's say we have $j = j_1$ and $j = j_2$. Then, it turns out that the total j is given by:

$$(j_1 + j_2), (j_1 + j_2 - 1), \cdots, (|j_1 - j_2| + 1), |j_1 - j_2| \tag{12}$$

We will neither show you the proof nor ask you to prove this, but this makes roughly some sense for two reasons. First, if you have two vectors $\vec{L}^{(1)}$ and $\vec{L}^{(2)}$, then the maximum magnitude which their sum can be $|\vec{L}^{(1)}| + |\vec{L}^{(2)}|$ (i.e. when they are parallel and in same direction) and the minimum magnitude being $||\vec{L}^{(1)}| - |\vec{L}^{(2)}||$ (i.e. when they are anti-parallel). Therefore, it would not make any sense if the total j is bigger than $j_1 + j_2$ or less than $|j_1 - j_2|$. Second, $j = j_1$ has $(2j_1 + 1)$ dimensions and the $j = j_2$ has $(2j_2 + 1)$ dimensions. So, the total vector space is their product. On the other hand, $j = (j_1 + j_2)$ has $2(j_1 + j_2) + 1$ dimensions and $j = (j_1 + j_2 - 1)$ has $2(j_1 + j_2 - 1) + 1$ and so on. If you sum them up, up to $j = |j_1 - j_2|$, you get exactly $(2j_1 + 1)(2j_2 + 1)$, so dimensions are matching. For example, if the first particle has j =3/2 and the second 1, we have:

$$(2 \times \frac{3}{2} + 1)(2 \times 1 + 1) = 12 \tag{13}$$

and, their total js are $5/2(= 3/2 + 1)$, $3/2$, $1/2(= |3/2 - 1|)$. Therefore, we have

$$(2 \times \frac{5}{2} + 1) + (2 \times \frac{3}{2} + 1) + (2 \times \frac{1}{2} + 1) = 12 \tag{14}$$

So, they are equal. One can easily prove this equality in general case (i.e., arbitrary j_1 and j_2) rigorously. (**Problem 2.** Show this equality rigorously.)

So, what is the angular momentum addition good for? In an earlier article, we explained the concept of orbital angular momentum and spin angular momentum. When you actually measure angular momentum, you usually measure the total angular momentum. So, you need to know how to add the orbital angular momentum and the spin angular momentum. Of course, in this article, we assumed that we were adding the angular momenta of two different particles, but we can add the orbital angular momentum and the spin angular momentum of a single particle in exactly the same way. Physicists and chemists were baffled by the line spectrums of atoms, and the angular momentum addition clarified them.

Summary

- If the angular momentums of two particles are given by $\vec{L}^{(1)}$ and $\vec{L}^{(2)}$, and if we call $\vec{L} = \vec{L}^{(1)} + \vec{L}^{(2)}$, what would be the eigenvector of L^2 and L_z? This is the angular momentum addition problem in quantum mechanics.

- If you add $j = j_1$ and $j = j_2$, their total angular momentum j can be $(j_1 + j_2), (j_1 + j_2 - 1), \cdots, (|j_1 - j_2| + 1), |j_1 - j_2|$. If you count the dimensions, they come correctly, being the product of the dimensions of $j = j_1$ and $j = j_2$.