Another example of differential equations

Four-year old Ung-yong Kim was asked to solve the following differential equation by a Japanese mathematician in a Japanese TV show.

$$(1 + x^2)dy - adx = xydx \tag{1}$$

Let's solve it together, partly by yourself and partly by myself. First, the above expression implies:

$$\frac{dy}{dx} - \frac{x}{1 + x^2}y = \frac{a}{1 + x^2} \tag{2}$$

Now, let $y = e^f g$ where $f = f(x)$ and $g = g(x)$, and set

$$f' = \frac{1}{1 + x^2} \tag{3}$$

Then, you will get (**Problem 1.** Check this!)

$$e^f g' = \frac{a}{1 + x^2} \tag{4}$$

You can now use (3) to obtain f and plug this into (4). Then, you get (**Problem 2.** Check this!)

$$g' = \frac{a}{(1 + x^2)^{3/2}} \tag{5}$$

which implies:

$$g = \int \frac{adx}{(1 + x^2)^{3/2}} \tag{6}$$

You have already solved the above integration in our earlier article "Integration by substitution." Since you now know f and g, from $y = e^f g$ you get y. The answer is following: (**Problem 3.** Check this!)

$$y = ax + c\sqrt{1 + x^2} \tag{7}$$

for an arbitrary c.