Application of Gauss' law

In this article, we will use Gauss' law to obtain the electric field in two simple cases. In the first case, a uniform surface charge density σ (i.e. charge per unit area) is placed at whole xy plane (i.e. $z = 0$). See Fig.1 and Fig.2. Of course, we only show a part of xy plane, because we cannot draw something that is infinitely wide. Given this, we want to find out what the electric field due to the electric charge is. First, it is easy to see that the electric field must be parallel to z axis from the symmetry. It cannot certainly bend to x or y direction. Now, as in the figure, let's take a cylinder as a Gaussian surface, and calculate the flux. Notice that there is no contribution to the flux on the side of the cylinder as the electric field, having only z-component, cannot pierce the side of cylinder. On the other hand, there is contribution to the flux from the bottom and the top of the cylinder. If the top area of cylinder (i.e. which is also equal to the bottom of the cylinder) is A, and if the magnitude of the electric field is given by E, which should be common for both $z < 0$ and $z > 0$ from symmetry, the flux contribution from the top is given by EA, while the bottom is also given by EA making $2EA$ combined together. As the charge inside this Gaussian surface is σA, we conclude:

$$\epsilon_0(2EA) = \sigma A, \qquad \rightarrow \qquad E = \frac{\sigma}{2\epsilon_0} \tag{1}$$

Notice that this is independent from the distance to the charge.

Now, to the second example. Uniform line charge density λ (i.e. charge per unit length) is placed on an infinitely long straight line. What is the electric field at a place a distance r away? See Fig.3. and Fig.4. Here, we take a cylinder with height h and radius r as a Gaussian surface. Notice also that there is no contribution to the flux from bottom and top of the cylinder, as electric field is parallel to these portions of Gaussian surface. So, the

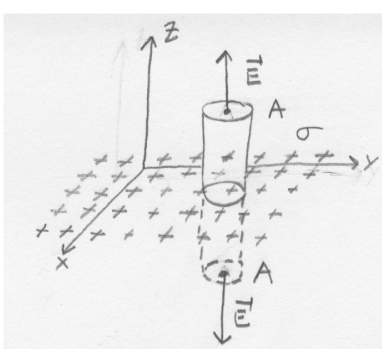

Figure 1: perspective view

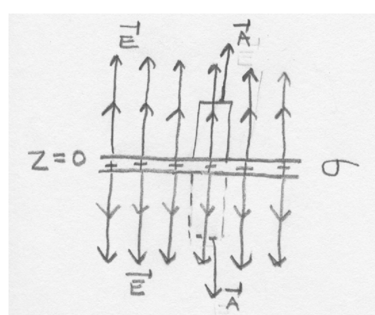

Figure 2: side view

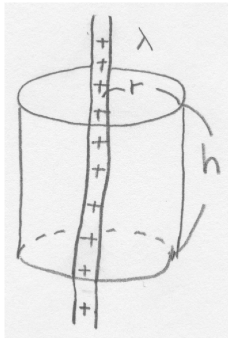

Figure 3: perspective view

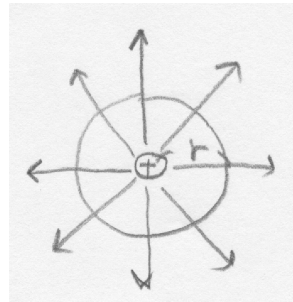

Figure 4: top view

contribution to the flux only comes from the side of the cylinder. As the electric field and the side of the cylinder are perpendicular to one another, the flux is simply given by

$$\int \vec{E} \cdot d\vec{A} = E(2\pi r h) \tag{2}$$

As the charge inside the Gaussian surface is λh, we conclude:

$$\epsilon_0 E(2\pi r h) = \lambda h, \qquad \rightarrow \qquad E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r} \tag{3}$$

Problem 1. Let's assume that the Earth is a uniform sphere with mass M and radius R. In other words, the density of the Earth is constant. (This is not true! We are just making this up to make the problem easier and solvable!) Then, let's say that you dig a straight tunnel from the North pole to the South pole that passes the center of the Earth. If you gently drop a box with mass m at North pole to the tunnel it will arrive at the South pole. Ignoring air resistance, how long will it take the box to reach there? (Hint[1])

Summary

- When there is a symmetry in the distribution of electric charge, it is sometimes possible to find the electric field using Gauss's law, which is a lot easier than using Coulomb's law which necessarily requires integration.

[1]Find the gravitational force at the distance r from the center. You can do it by taking a Gaussian surface by a sphere with radius r centered around the center of the Earth. Then, as far as the gravitational force is concerned, you can consider as if all the mass in the Gaussian surface is located at the center. If you do it correctly, the gravitational force will be proportional to r. So, this becomes a problem for harmonic oscillator.