# Application of residue theorem

In this article, we will show you how we can apply residue theorem to a certain type of integration. To this end, we will closely follow Wikipedia.

Suppose we want to calculate the following:

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx \tag{1}$$

Assuming $t \geq 0$ (we will consider the other case later), consider the following integral, to this end:

$$\int_C \frac{e^{itz}}{z^2+1}\,dz \tag{2}$$

where the contour $C$ is drawn in Fig.1. The contour goes along the real line from $-a$ to $a$ and then counter-clockwise along a semicircle from $a$ to $-a$. Notice that there is a pole when $z = i$ and $z = -i$ as $z^2 + 1$, the denominator of the integrand becomes zero in such cases. As only the pole at $z = i$ is encompassed by the contour, we only need to consider the residue at $z = i$ when calculating the integral. We get:

$$\int_C \frac{e^{itz}}{z^2+1}\,dx = \int_C \frac{e^{itz}/(z+i)}{z-i}\,dz = 2\pi i \frac{e^{-t}}{2i} = \pi e^{-t} \tag{3}$$

The contour consists of the straight part and the arc part. Therefore, we get:

$$\int_{\text{straight}} \frac{e^{itz}}{z^2+1}\,dz + \int_{\text{arc}} \frac{e^{itz}}{z^2+1}\,dz = \pi e^{-t} \tag{4}$$

In other words,

$$\int_{-a}^{a} \frac{e^{itz}}{z^2+1}\,dz = \pi e^{-t} - \int_{\text{arc}} \frac{e^{itz}}{z^2+1}\,dz \tag{5}$$

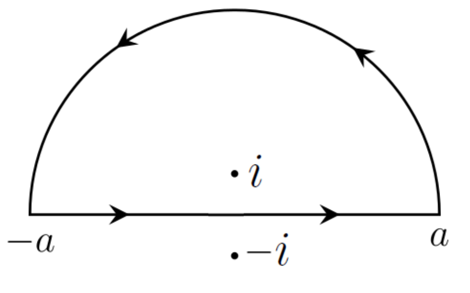


Figure 1: $t \geq 0$

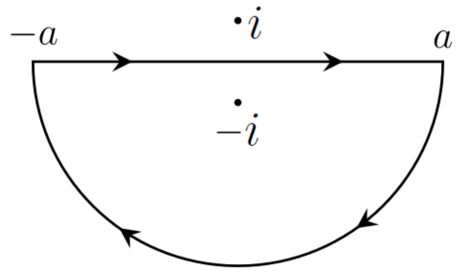


Figure 2: $t \leq 0$

Now we will show that the arc part goes to zero in the limit $a$ goes to infinity. To show this, first notice the following, which is satisfied on the arc when $t \geq 1$:

$$\left|e^{itz}\right| = \left|e^{it|z|(\cos\phi + i\sin\phi)}\right| = \left|e^{-t|z|\sin\phi + it|z|\cos\phi}\right| = e^{-t|z|\sin\phi} \leq 1. \tag{6}$$

where we have used $\sin\phi \geq 0$ satisfied on the arc. Then, we have:

$$\left|\int_{\text{arc}} \frac{e^{itz}}{z^2+1}\,dz\right| \leq \int_{\text{arc}} \left|\frac{e^{itz}}{z^2+1}\right| dz \leq \int_{\text{arc}} \frac{1}{|z^2+1|} dz \leq \int_{\text{arc}} \frac{1}{a^2-1} dz = \frac{\pi a}{a^2-1} \tag{7}$$

as the right-hand side goes to zero in the limit $a$ goes to infinity, we conclude that the arc part is zero in such a limit. Therefore, from (5), we obtain:

$$\lim_{a\to\infty} \int_{-a}^{a} \frac{e^{itz}}{z^2+1}\,dz = \pi e^{-t} \tag{8}$$

from which we conclude:

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx = \pi e^{-t} \tag{9}$$

Now, to the case when $t \leq 0$. It turns out that in such a case we have to consider the contour shown in Fig.2. (We will see why soon.) As the contour is clock-wise, we get an extra negative sign. We get:

$$\int_C \frac{e^{itz}}{z^2+1}\,dx = \int_C \frac{e^{itz}/(z-i)}{z+i}\,dz = -2\pi i \frac{e^t}{-2i} = \pi e^t \tag{10}$$

Again, we can split the contour into the straight part and the arc part, and the arc part becomes zero as well, since the following is satisfied when $t \leq 0$ and $\sin\phi \leq 0$

$$\left|e^{itz}\right| = \left|e^{it|z|(\cos\phi + i\sin\phi)}\right| = \left|e^{-t|z|\sin\phi + it|z|\cos\phi}\right| = e^{-t|z|\sin\phi} \leq 1. \tag{11}$$

and (7) is satisfied. Therefore, we conclude:

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx = \pi e^{t} \tag{12}$$

when $t \leq 0$. Using the following Heaviside step function,

$$\theta(x) = 0 \;\text{ if } x < 0, \qquad \theta(x) = 1 \;\text{ if } x \geq 0 \tag{13}$$

it can be represented as:

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx = \pi e^{-t}\theta(t) + \pi e^{t}\theta(-t) \tag{14}$$

**Problem 1.** Evaluate the following. (Hint[1])

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} \tag{15}$$

[1] Use the result of Problem 8 in "Euler's formula and hyperbolic functions."

**Problem 2.** Prove the following, which we will use in our later article "The Feynman propagator of the scalar field."

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dx}{x^2-E^2+i\epsilon}e^{ixt}=-\frac{e^{-iEt}\theta(t)+e^{iEt}\theta(-t)}{2E} \tag{16}$$

where $\epsilon$ is an infinitesimal, but positive number and $E>0$. (Hint: Show that the pole is at $x=-E+i\epsilon'$ and $x=E-i\epsilon'$, where $\epsilon'$ is another infinitesimal, but positive number given by $\epsilon'=\epsilon/(2E)$. The rest is exactly same as our example in this article.)

## Summary

- Sometimes, the residue theorem can be used to calculate the integration of following form

  $$\int_{-\infty}^{\infty}f(x)dx$$

  by turning it into a contour integral, if

  $$\int_{\text{arc}}f(x)dx$$

  can be shown to approach zero, where "arc" is the infinite half circle (either clockwise or anti-clockwise or both) starting from $+\infty$ to $-\infty$.