Area inside a circle, surface area of a sphere, and volume inside a sphere

Let's apply calculus to obtain the area inside a circle, surface area of a sphere and volume inside a sphere.

Let's first obtain the area inside a circle with radius R. See Fig.1. We can regard the region inside a circle as the collection of bands with infinitesimal (i.e. infinitely small) width. One of such bands is shaded in the figure. It has length $2\pi r$ and width dr. As the width is very small, we can regard it as a bent rectangle with the length $2\pi r$ and the width dr. Therefore, the band has area $2\pi r dr$. Now, we are ready to calculate the area inside the circle. It is the sum of the area of each band. As r runs from 0 to R, we have:

$$A_{\rm circle} = \int_0^R 2\pi r dr = \pi R^2 \tag{1}$$

Let's move on to our second example. See Fig.2. We can regard the sphere as the collection of bands with infinitesimal width. One of such bands is drawn in the figure. It has length $2\pi R \sin \theta$ and width $Rd\theta$. As the width is very small, we can regard it as a bent rectangle with the length $2\pi R \sin \theta$ and the width $Rd\theta$. Therefore, the band has area $2\pi R^2 \sin \theta d\theta$. As the area of the sphere is the sum of the area of each band, and as θ runs from 0 to π , we have:

$$A_{\rm sphere} = \int_0^{\pi} 2\pi R^2 \sin\theta d\theta = 2\pi R^2 \left(-\cos\pi - (-\cos0) \right) = 4\pi R^2 \tag{2}$$

Now, our third example. See Fig.3. We want to calculate the volume inside a sphere with radius R. We can regard the region inside the sphere as a sum of discs. The volume inside the sphere will be the sum of each disc. One of such discs is denoted in the figure. The disc has radius $\sqrt{R^2 - h^2}$ from Pythagorean theorem and width dh, where h varies from -R to





Figure 1: area inside a circle with radius R

Figure 2: area of a sphere with radius R





Figure 3: volume inside a sphere with radius ${\cal R}$

Figure 4: volume of a right circular cone with radius r and height h

R. The volume of each disc is given by $dV = \pi (\sqrt{R^2 - h^2})^2 dh$. Therefore, the total volume is given as follows:

$$V = \int_{-R}^{R} \pi (\sqrt{R^2 - h^2})^2 dh = \pi \left(R^2 h - \frac{h^3}{3} \right) \Big|_{-R}^{R} = \frac{4}{3} \pi R^3$$
(3)

In our later article "Spherical coordinate system, the surface area of sphere, and the volume of ball" we will obtain the volume inside a sphere again by a similar, but different method using trigonometric functions.

Final comment. Ancient Greek mathematician and physicist Archimedes was the first one who obtained the area inside a circle, the surface area of a sphere, the volume inside a sphere and the volume of cone. Of course, as this was about two thousand years before calculus was invented, he didn't use calculus to obtain them but had to use geometric methods.

Problem 1. What is the volume of a right circular cone with radius r and height h? See Fig.4. (Hint¹)

Summary

• Area and volume of various objects can be obtained by integration.

 $^{^1\}mathrm{Regard}$ the cone as the sum of discs. Think about what the radius of each disc is.