Area of a sphere

In an earlier article, we obtained the area of a sphere using Archimedes's method. In this article, we will obtain the area of a sphere by using another method. We can do so, as we have obtained the volumes of cones and spheres in the last article. However, before doing so, it is instructive to recall how we obtained the area of a circle in our earlier article.

In Archimedes's method, we approximated a circle as a regular polygon, which we then divided into triangles to calculate its area. The area of each triangle is given by

$$A_{\text{triangle}} = \frac{1}{2}sh \tag{1}$$

where s is each side of polygon and h the height. In Archimedes's case, we saw that h goes to r, as the number of sides get bigger and bigger. Then, the area of a circle is given by

$$A_{\text{circle}} = \sum_{\text{triangles}} A_{\text{triangle}} = \sum_{\text{triangles}} \left(\frac{1}{2}sr\right) = \frac{1}{2}\left(\sum s\right) r = \frac{1}{2}(2\pi r)r = \pi r^2 \tag{2}$$

Notice here that, in the second to last equality, we used the fact that the total sum of the sides is given by the circumference of the circle in the limit the number of sides of the polygon get larger and larger.

In the second method, we divided a circle into very thin sectors, and used the fact that each sector can be regarded as a "triangle" with base its arc, and its height the radius of the circle. Again, we used the fact that the total sum of the base is the circumference of the circle.

In other words, the area of a circle is equal to the area of triangle with the base being the circumference of the circle and the height the radius of the circle.

Then, what should be the volume of a sphere? How can we connect it with the area of a sphere? If you think about this a little bit, you will notice that it should be equal to the volume of cone with base the area of sphere and height the radius of sphere.

Think this way. You can divide the surface of a sphere into very small segments. It doesn't really matter how you divide it, as long as each segment is very very small. See Fig. 1. If you connect each segment with the center of the sphere you get a cone. See Fig. 2. The volume of the cone is given by

$$V_{\text{cone}} = \frac{1}{3}Ar \tag{3}$$

where A is the area of the segment on the surface of the sphere. Then, as the volume of the

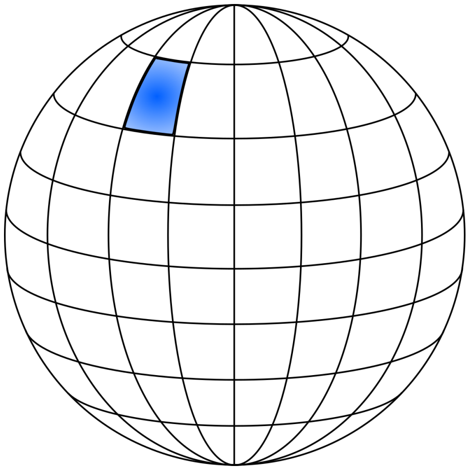

Figure 1: Wireframe representation of a sphere. The vertical and horizontal curves correspond to meridian and longitudinal lines, respectively. However, it doesn't matter how the wireframes look like as long as they are dense.

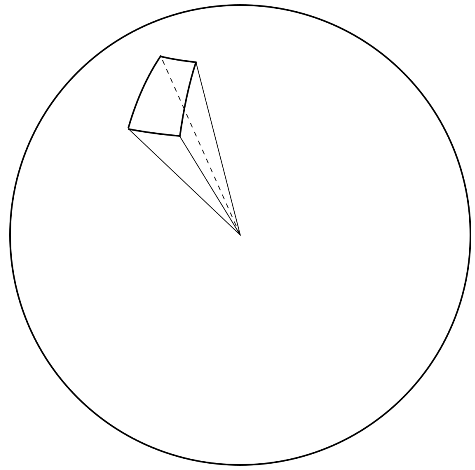

Figure 2: Cone with a small rectangular base. The cone's vertex is the sphere's center, while its base is one of the wireframes' tiles marked blue in Fig. 1.

sphere is the total sum of these cones, we have

$$V_{\text{sphere}} = \sum_{\text{segments}} V_{\text{cone}} = \sum_{\text{segments}} \frac{1}{3} A r = \frac{1}{3} \left(\sum A \right) r = \frac{1}{3} A_{\text{sphere}} r \tag{4}$$

From the last article, we know that the volume of a sphere is

$$V_{\text{sphere}} = \frac{4}{3} \pi r^3 \tag{5}$$

Thus, we conclude

$$A_{\text{sphere}} = V_{\text{sphere}} / (r/3) = 4\pi r^2 \tag{6}$$

Final comment. This is not the derivation that I first learned of the area of a sphere; I first learned a derivation that uses calculus. However, I later re-discovered this derivation during my musing when I was a freshman at Korea Advanced Institute of Science and Technology. I realized the relation between the volume and the area of sphere. Then, I used these relations to go on to calculate the volume and area of higher-dimensional spheres and volumes. Of course, I was sure that somebody had figured out my derivation several centuries ago, but it was fun.

Summary

- The volume of a sphere is given by the volume of a cone with its base the area of sphere and the height the radius of sphere.
- The area of a sphere with radius r is $4\pi r^2$.