Area of a sphere by Archimedes' method

In this article, we will introduce how Archimedes calculated the area of a sphere. See Fig. 1. A sphere is fit in a cylinder. If the radius of the sphere is r, the base of the cylinder is a circle with radius r, and the height of the cylinder is $2r$. What Archimedes showed was that the area of the sphere is equal to the area of the side of the cylinder.

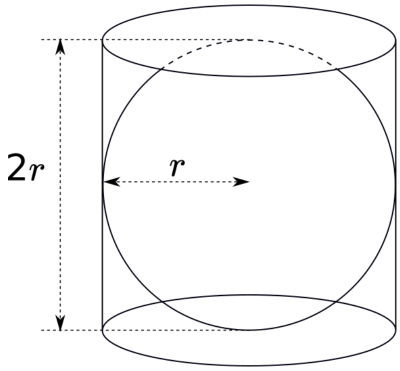

Figure 1: The sphere is exactly contained inside a cylinder.

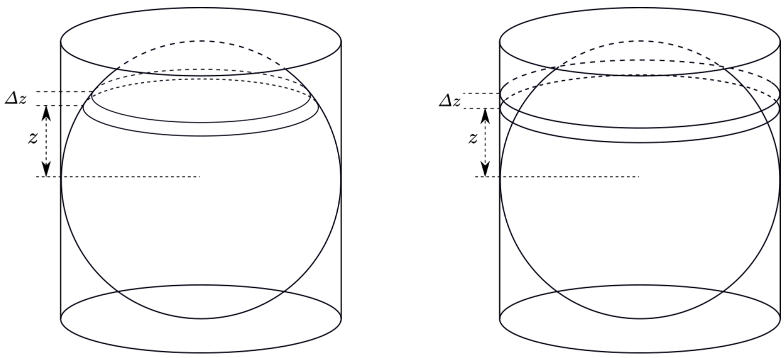

Figure 2: The same sphere and cylinder of Fig. 1 but with a stripe drawn. On the left subfigure, the stripe is drawn around the sphere, while on the right subfigure the stripe is around the cylinder.

How did he show this? See Fig. 2. On the left, you see a stripe on a sphere. This stripe is located z meter above the center of sphere, and when projected on the cylinder, it has a vertical width Δz. We will assume that this Δz is very small. On the right, you see a stripe on a cylinder. This stripe is located at the same vertical position as the left one; it is also located z meter above the center of sphere, and has a width Δz. As this stripe has length $2\pi r$, its area is

$$A_{\text{stripe}} = 2\pi r \Delta z \tag{1}$$

What Archimedes showed was that the stripe on the left and the stripe on the right have the same area.

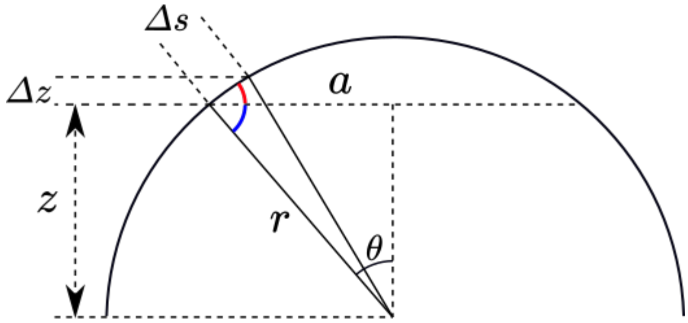

Figure 3: Side view of Fig. 2.

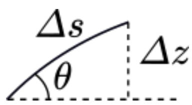

Figure 4: $\Delta z = \Delta s \sin\theta$

Let's see why. See Fig. 3 for the side view of Fig. 2. The width of the stripe is now Δs. As the length of the stripe is $2\pi a$, where $a = r\sin\theta$, the area of the stripe is

$$A_{\text{stripe}} = \Delta s \times 2\pi a = \Delta s \times 2\pi r \sin\theta = 2\pi r \Delta s \sin\theta \tag{2}$$

However, we can express this quantity in terms of Δz. Notice that the angle marked in blue is $90^\circ - \theta$. Therefore, the angle marked in red is θ. Now, see Fig. 4 for a close view of this part of Fig. 3. We see

$$\Delta z = \Delta s \sin\theta \tag{3}$$

You may want to argue that this formula is not exactly satisfied, as the arc is curved, but if we choose small enough Δs and Δz, the curvedness of the arc can be neglected. Then, the arc can be considered as having a constant slope. This makes (3) an *exact* equation. Plugging this back into (2), we recover (1). The area of the two stripes are indeed equal.

Given this, see Fig. 5. Imagine that we divide the sphere by such very thin stripes. Then, each stripe has its counterpart in the cylinder. As each pair of stripes has the same area, the area of the sphere, which is the sum of area of the stripe, is equal to the area of the side of the cylinder, which is $2\pi r \times 2r$. Thus, the area of a sphere is

$$A = 2\pi r \times 2r = 4\pi r^2 \tag{4}$$

Problem 1. (Very challenging!) In our earlier article "Manifold," we introduced the concept of n-sphere. In this article, we saw that the (2-dimensional) area of S^2 (i.e.,

Figure 5: A sphere and a cylinder with colored stripes. Each equal-colored stripe has the same area.

2-sphere) with radius r is $4\pi r^2$, by using Archimedes method. By applying this method, figure out the (3-dimensional) volume of S^3 (i.e., 3-sphere) with radius r. (Hint[1])

Summary

- The area of sphere with radius r is equal to $4\pi r^2$.

[1] Notice that Archimedes related the area of 2-sphere with the length of 1-ball. Try to find a similar connection.