# Area operator in terms of newer variables and dreibein

## 1 Area operator expressed in terms of metric

In this article, we will show how the area operator, which plays a key role in loop quantum gravity, can be expressed in terms of the metric. In particular, we will express it in terms of newer variables by using a hand-waving argument. Then we will express the area operator in terms of "dreibein" which we show by somewhat hand-waving argument again. (Drei is "three" in German, just as "Vier" in Vierbein is "four.") Then we will derive the expression for the area operator in terms of newer variables again, this time, rigorously.

Our convention in this article is as follows: we use $i$, $j$, $k$ for Lorentz indices and $a$, $b$, $c$, $d$, $e$, $f$ for space indices. We denote the Levi-Civita symbol using tilde as in $\tilde{\epsilon}_{123} \equiv 1$, while the Levi-Civita tensor is written without tilde as in $\epsilon_{123} \equiv \sqrt{g}$.

The length of a curve in Euclidean space is given as follows:

$$L = \int \sqrt{dx^2 + dy^2 + dz^2} \tag{1}$$

Similarly, the area of a surface in Euclidean space is given as follows:

$$A = \int \sqrt{(dx \wedge dy)^2 + (dy \wedge dz)^2 + (dz \wedge dx)^2} \tag{2}$$

We will not prove this formula but merely hope that the reader finds it reasonable; when the surface concerned lies on the $x$-$y$ plane, it reduces to $A = \int dx \wedge dy$ (and similarly for the cases of the $y$-$z$ and $z$-$x$ planes). If the reader doesn't find this expression reasonable, he or she can rely on the rigorous treatment at the end of this article.

Now let's define:

$$E^1 = dy \wedge dz, \qquad E^2 = dz \wedge dx, \qquad E^3 = dx \wedge dy \tag{3}$$

Then it is easy to see the following:

$$A = \int (E^1)^2 + (E^2)^2 + (E^3)^2 = \int \sum_i E^i E^i \tag{4}$$

Given these definitions, let's introduce primed coordinate as follows, for which the metric is diagonal and constant

$$dx = \sqrt{g_{11}} dx', \qquad dy = \sqrt{g_{22}} dy', \qquad dz = \sqrt{g_{33}} dz' \tag{5}$$

Then, we have:

$$E^1 = \frac{\sqrt{g}}{\sqrt{g_{11}}} dy' \wedge dz', \qquad E^2 = \frac{\sqrt{g}}{\sqrt{g_{22}}} dz' \wedge dx', \qquad E^3 = \frac{\sqrt{g}}{\sqrt{g_{33}}} dx' \wedge dy' \tag{6}$$

Therefore, if we write:

$$E^i = \frac{1}{2} D^{ia} \epsilon_{abc} dx'^b \wedge dx'^c \tag{7}$$

where $\epsilon_{abc}$ is the Levi-Civita tensor defined by $\epsilon_{123} = \sqrt{g}$, we have:

$$D^{11} D^{11} = \frac{1}{g_{11}} = g^{11} \tag{8}$$

and similarly for $D^{22}$ and $D^{33}$. One can also show that in our case, $D^{ia}$ is diagonal. Therefore, we can conclude:

$$\sum_i D^{ia} D^{ib} = g^{ab} \tag{9}$$

Now, to dreibein. We use the following notation:

$$e^i = e^i_a dx^a \tag{10}$$

where the Einstein summation convention is used. Notice that dreibeins are invariant under the change of coordinates in (5), since $e^i_a$ transforms as a co-vector, while $dx^a$ transforms as a vector. (Remember that the product of vector and co-vector must be a scalar, which doesn't depend on the choice of coordinate.) This should indeed be true for any coordinate transformation since $e^i$ is a scalar as far as space indices are concerned, as space indices are absent.

Given this, notice that the following is true for Euclidean space:

$$E^1 = e^2 \wedge e^3, \qquad E^2 = e^3 \wedge e^1, \qquad E^3 = e^1 \wedge e^2 \tag{11}$$

Since $E$s don't depend on the choice of coordinate, as $e$s don't, it should be reasonable that the above formula is correct, if the space concerned is Euclidean space. However, it turns out that it is true for any space. This is especially reasonable when you remember that $e^1 \wedge e^2 \wedge e^3$ is the invariant volume form regardless of any coordinate choice.

Finally, we show the rigorous derivation. To this end, recall that volume is given as follows:

$$V = \int d^3x \sqrt{\det g} \tag{12}$$

In our case, what we need is not volume, but area, which is two-dimensional. Let's say that $\tau$ and $\sigma$ are the two-dimensional coordinates that parametrize the area, and let's use Greek indices $\alpha$ and $\beta$ to denote them. In other words, $\alpha = 0$ is $\tau$ and $\alpha = 1$ is $\sigma$. Then, the analogous equation to (12) is:

$$A = \int d\tau d\sigma \sqrt{\det G_{\alpha\beta}} \tag{13}$$

Now, let's calculate the metric $G_{\alpha\beta}$. The line element in 3-d space must match with the line element in $\tau$ and $\sigma$ variables. This implies:

$$ds^2 = g_{ab}dx^a dx^b = g_{ab}\frac{\partial x^a}{\partial \alpha}\frac{\partial x^b}{\partial \beta}d\alpha d\beta = G_{\alpha\beta}d\alpha d\beta \tag{14}$$

which implies:

$$G_{\alpha\beta} = \frac{\partial x^a}{\partial \alpha}\frac{\partial x_a}{\partial \beta} \tag{15}$$

Therefore, (13) can be re-written as:

$$A = \int d\tau d\sigma \sqrt{\det\left[\frac{\partial x^a}{\partial \alpha}\frac{\partial x_a}{\partial \beta}\right]} \tag{16}$$

$$= \int d\tau d\sigma \left(\det\begin{bmatrix} \frac{\partial x^a}{\partial \tau}\frac{\partial x_a}{\partial \tau} & \frac{\partial x_d}{\partial \tau}\frac{\partial x^d}{\partial \sigma} \\ \frac{\partial x_c}{\partial \sigma}\frac{\partial x^c}{\partial \tau} & \frac{\partial x^b}{\partial \sigma}\frac{\partial x_b}{\partial \sigma}\end{bmatrix}\right)^{1/2} \tag{17}$$

which means:

$$A = \int \left[\frac{\partial x^a}{\partial \tau}\frac{\partial x^b}{\partial \sigma}\frac{\partial x^c}{\partial \tau}\frac{\partial x^d}{\partial \sigma}(g_{ac}g_{bd} - g_{ad}g_{bc})\right]^{1/2} d\tau d\sigma \tag{18}$$

This equation implies:

$$A = \int \left[\frac{\partial x^a}{\partial \tau}\frac{\partial x^b}{\partial \sigma}\frac{\partial x^c}{\partial \tau}\frac{\partial x^d}{\partial \sigma}(\epsilon_{eab}\epsilon_{fcd}g^{ef})\right]^{1/2} d\tau d\sigma \tag{19}$$

Now, using (7), (4) can be re-expressed as follows:

$$A = \int \left[\sum_i (\frac{1}{2}D^{ei}\epsilon_{eab}dx^a dx^b)(\frac{1}{2}D^{fi}\epsilon_{fcd}dx^c dx^d)\right]^{1/2}$$

$$= \int \left[\sum_i D^{ei}\epsilon_{eab}\frac{\partial x^a}{\partial \tau}\frac{\partial x^b}{\partial \sigma}d\tau d\sigma D^{fi}\epsilon_{fcd}\frac{\partial x^c}{\partial \tau}\frac{\partial x^d}{\partial \sigma}d\tau d\sigma\right]^{1/2} \tag{20}$$

Comparing this formula with (19), we conclude (9).

## 2 "New" gravitational electric field as area operator

In 1993, Rovelli asserted that, if the area two form is given by

$$E^i(x) = E^{ia}(x)\tilde{\epsilon}_{abc}dx^b \wedge dx^c \tag{21}$$

then the norm of $E^{ia}$ gives the area spectrum. Here $E^{ia}$ is called the "gravitational electric field" and is defined by:

$$gg^{ab} = \sum_i E^{ia}E^{ib} \tag{22}$$

However, Rovelli's derivation was wrong. The formula he meant to work with is:

$$E^i(x) = \frac{1}{2}E^{ia}(x)\tilde{\epsilon}_{abc}dx^b \wedge dx^c \tag{23}$$

but he used the following incorrect relation:

$$\int \epsilon_{eab} dx^a \wedge dx^b = \int \epsilon_{eab} \frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \tau} d\sigma d\tau \tag{24}$$

Let's check this! Suppose $e = 1$, $x^2 = \sigma$, $x^3 = \tau$. The left-hand side becomes

$$\int \epsilon_{123} dx^2 \wedge dx^3 + \epsilon_{132} dx^3 \wedge dx^2 = \int 2dx^2 \wedge dx^3 \tag{25}$$

while the right-hand side becomes

$$\int (\epsilon_{123} \frac{\partial \sigma}{\partial \sigma} \frac{\partial \tau}{\partial \tau} + \epsilon_{132} \frac{\partial \tau}{\partial \sigma} \frac{\partial \sigma}{\partial \tau}) d\sigma d\tau = \int d\sigma d\tau = \int dx^2 dx^3 \tag{26}$$

They are not equal. We conclude:

$$\int \epsilon_{eab} \frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \tau} d\sigma d\tau = \int \frac{1}{2} \epsilon_{eab} dx^a \wedge dx^b \tag{27}$$

Actually, (23) is not correct. Remember that you raise and lower space indices by $g^{ab}$ and $g_{ab}$ and you raise and lower Lorentz indices with $\eta^{ij}$ and $\eta_{ij}$. In (23), one should use the Levi-Civita tensor $\epsilon_{123} \equiv \sqrt{g}$ because $a$, $b$, $c$ are not Lorentz indices but space indices. Therefore, it is not appropriate to use $E^{ia}(x)$ in (23). For example, in string theory, the action of a string in the presence of background fields is given as follows:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} [(g^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X)) \partial_a X^\mu \partial_b X^\nu + \alpha' R \Phi(X)] \tag{28}$$

where $\epsilon^{ab}$ is not a Levi-Civita symbol but a Levi-Civita tensor.

Another way of seeing that it is reasonable to use the Levi-Civita tensor is that the formula for the hodge dual always includes $\sqrt{g}$ (i.e. the Levi-Civita tensor) as follows:

$$(dx_a)^* = \frac{1}{2!} \epsilon_{abc} dx^b \wedge dx^c \tag{29}$$

Therefore, we suggest:

$$E^i(x) = \frac{1}{2} D^{ia}(x) \epsilon_{abc} dx^b \wedge dx^c \tag{30}$$

In the last section, we have shown the following must be satisfied:

$$g^{ab} = \sum_i D^{ia} D^{ib} \tag{31}$$

This suggests that $D$ should be dreibein: $D^{ia} = e^{ia}$. In conclusion, we have:

$$E^i(x) = \frac{1}{2} e^{ia}(x) \epsilon_{abc} dx^b \wedge dx^c \tag{32}$$

To obtain a theory that uses this "new" gravitational electric field instead of the traditional gravitational electric field (i.e. Ashtekar variables), we will introduce "newer" variables in the next article.

## Summary

- An area element can be written as

$$E^i(x) = \frac{1}{2} e^{ia}(x) \epsilon_{abc} dx^b \wedge dx^c$$

  where $\epsilon_{abc}$ is a Levi-Civita tensor.