Asymptotic behavior of polynomials

Consider the following polynomial:

$$f(x) = \frac{x}{100} - 100x^2 + 1000x^3 + \frac{x^4}{1000} \tag{1}$$

Now, let's ask the following two questions:

- Which of the four terms is most important when x is very big?
- Which of the four terms is most important when x is very small (i.e. close to 0)?

To answer the first question, let's plug in a big number, say, 1000. Then, the first term is 10, the second term, -10^8, the third term 10^{12}, the fourth term 10^9. Therefore, you may think the third term is most important as other terms are negligible compared to the third term 10^{12}. You may think so, because the coefficient in front of x^3 has a large number 1000, while the coefficient in front of x^4 is very small, being $\frac{1}{1000}$. However, if you choose a even larger number, the fourth term is much bigger than the third term. It's because the ratio of the third term to the fourth term can be arbitrarily small if you choose a very big x. Let's see what I mean. The ratio is given by

$$\frac{1000x^3}{x^4/1000} = \frac{1000000}{x} \tag{2}$$

If you choose a moderately big number such as $x = 1000$, the ratio is still big, being 1000. However, if you choose a number much bigger than 1000000, say, $x = 10^{10}$, the ratio is 0.0001 being really small. Similarly, the ratios of the first term and the second term to the fourth term are very small for big x. Therefore, we conclude that $x^4/1000$ is the most important term for large enough x. We say that $x^4/1000$ is the "leading order" term of $f(x)$ for large x.

Now, to answer the second question, let's plug in a small number, say 0.01. Then, the first term is 0.0001, the second term is -0.01, the third term 0.001, the fourth term 0.00000000001. Therefore, you may think that the second term is most important as other terms are negligible compared to the second term -0.01. However, if you choose a even smaller number, the second term can be negligible compared to the first term. To see this, let's calculate the ratio just as before. We have

$$\frac{-100x^2}{x/100} = -10000x = -\frac{x}{0.0001} \tag{3}$$

If you choose a moderately small number such as $x = 0.01$ the ratio is still not close to 0, being -100. However, if you choose a number much smaller than 0.0001, the ratio can be arbitrarily close to 0. The same can be said for the third term and the fourth term. They are negligible when x is very small. Therefore, we conclude that $x/100$ is the most important term for small enough x. We say that $x/100$ is the "leading order" term of $f(x)$ for small x.

What we said can be easily generalized. For example, for

$$g(x) = 3x^2 + 4x^3 - 3x^4 + 5x^8 \tag{4}$$

$5x^8$ is the leading order term for large x as it is the highest order (being 8) term, while $3x^2$ is the leading order term for small x as it is the lowest order (being 2) term. This is the asymptotic (i.e. for large x and for small x) behavior of polynomials.

Problem 1. Which term in $h(x)$ is most important when x is very small (i.e. close to 0), and when x is very big?[1]

$$h(x) = -\frac{5}{x^2} + \frac{1}{x} + \frac{3}{\sqrt{x}} \tag{5}$$

If you read our essay "The imagination in mathematics: "Pascals triangle, combination, and the Taylor series for square root"," you can challenge yourself to solve the following problems.

Problem 2. In thar article, we have seen that

$$\sqrt{1+x} = 1 + x/2 + \mathcal{O}(x^2) \tag{6}$$

where $\mathcal{O}(x^2)$ is a term of order x^2 or higher.

Use

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots \tag{7}$$

and $\cos\theta = \sqrt{1-\sin^2\theta}$, to show the following:

$$\cos\theta = 1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4) \tag{8}$$

where $\mathcal{O}(\theta^4)$ is a term of order θ^4 or higher.

[1] Of course, $h(x)$ is no longer a polynomial, but we can approach this problem in the same way as we learned in this article

Actually, if you calculate higher order terms, you will be able to check

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots \tag{9}$$

even though there is much an easier way to derive this as you will see in our later article "Taylor series."

Problem 3. From (7), (8) and $\tan\theta = \sin\theta/\cos\theta$, check

$$\tan\theta = \theta + \frac{\theta^3}{3} + \mathcal{O}(\theta^5) \tag{10}$$

Problem 4. In our earlier article "Parallax, how do we determine distances to stars?" we promised to invite you to check how good approximation $\tan\theta \approx \theta$ is, when θ is the annual parallax of a "near" star.[2] As you can read in that article, the annual parallax of 61 Cygni, the star of which the annual parallax was first noticed and measured, is about 0.3 arcsecond. (Recall that 60 arcsecond=1 arcminute, 60 arcminute=1 degree.) Now, look at the right-hand side of (10) carefully again. By checking the ratio of the second term to the first term, when θ is 0.3 arcsecond, explain why using $\tan\theta$ instead of θ to calculate the distance to 61 Cygni is useless, unless we can measure θ at least better than 1 part in 10^{12}. As mentioned in that article, currently θ can be measured only within the accuracy of 1 part in 5000 (i.e., 0.02%).

Summary

- Let's say we have a polynomial of x. Then, the most important term is the highest order term for large x, and the lowest order term for small x (i.e., x close to zero).
- The most important term is often called the leading order term.

[2]This is ironic as the annual parallax is so small for even the nearest star.