Bosons, Fermions in Quantum Mechanics picture

In two earlier articles, I introduced what bosons and fermions are. However, the formalism was that of the quantum field theory picture, in which Grassmann numbers are natural objects. In this article, I will discuss how bosons and fermions are interpreted in the quantum mechanics picture, which does not use Grassmann numbers.

In quantum mechanics, one always talks about wave functions. In the case of a system with just one particle, we can associate thereto a wave function $\psi(x)$. Roughly speaking, it has the property that the square of the absolute value of the wave function at x, is the probability that one will find the corresponding particle at the position x. In other words, we have $|\psi(x)|^2$ chance of finding the particle at the position x.[1]

There is a natural extension of the wave function to two or more particles. For example, $|\psi(x_1, x_2)|^2$ is the probability that one will find the first particle at x_1, and the second particle at x_2. Now comes the main story. Let's consider the case in which the first particle is the same as the second particle. Physicists call such particles "identical particles." Here, one cannot distinguish the case in which the first particle is at the position x_1 and the second particle at the position x_2 from the case in which the first particle is at the position x_2 and the second particle at the position x_1. Since both are substantially the same case, the probability for each case should be the same. Therefore, we can conclude: $|\psi(x_2, x_1)|^2 = |\psi(x_1, x_2)|^2$

This implies that $|\psi(x_2, x_1)| = |\psi(x_1, x_2)|$ which in turn implies:

$$\psi(x_2, x_1) = c\psi(x_1, x_2) \tag{1}$$

where c is a constant that doesn't depend on x_1 and x_2. Now, if we change the variables x_1 and x_2 to the variables x_2 and x_1 we get:

$$\psi(x_1, x_2) = c\psi(x_2, x_1) \tag{2}$$

Plugging (1) to (2), we get $\psi(x_1, x_2) = c^2\psi(x_1, x_2)$. As $\psi(x_1, x_2) = \psi(x_1, x_2)$, we conclude that $c^2 = 1$. This implies that c is either 1 or -1. When c is 1, we call the corresponding particles bosons, while we call the corresponding

[1] Precisely speaking, this is not the probability but the probability density, which we will introduce in our later article "Probability density function."

particles fermions, when c is -1. In other words:

$$\psi(x_2, x_1) = \psi(x_1, x_2) \text{ for bosons}$$
$$\psi(x_2, x_1) = -\psi(x_1, x_2) \text{ for fermions}$$

What happens when $x_2 = x_1$? In case of bosons, this doesn't give any new information as $\psi(x_1, x_1) = \psi(x_1, x_1)$ However, in case of fermions, this implies $\psi(x_1, x_1) = -\psi(x_1, x_1)$ which means that $\psi(x_1, x_1) = 0$. Therefore, two or more identical fermions cannot occupy the same position. In such cases, the wave function is zero, which means the probability is zero. This is called Pauli's exclusion principle, as explained in the earlier article.

Problem 1. If a wave function of three identical particles satisfy the following:

$$\psi(x_1, x_2, x_3) = \psi(x_2, x_3, x_1)$$

Can we conclude that the concerned particles are bosons? Why or why not? (Hint[2])

Summary

- For a wave function $\psi(x_1, x_2)$, $|\psi(x_1, x_2)|^2$ is the probability that one will find the first particle at x_1, and the second particle at x_2.
- We have $\psi(x_2, x_1) = \psi(x_1, x_2)$ for bosons, and $\psi(x_2, x_1) = -\psi(x_1, x_2)$ for fermions.
- As we have $\psi(x_1, x_1) = 0$ for fermions, two or more identical fermions cannot occupy the same state.

[2] Use the relations $\psi(x_1, x_2, x_3) = \pm\psi(x_2, x_1, x_3)$ and $\psi(x_1, x_2, x_3) = \pm\psi(x_1, x_3, x_2)$ where + sign is for bosons and − sign is for fermions. Or, you can think about whether this is an even permutation or an odd permutation.