Bosons, Fermions, and Pauli's exclusion principle

According to physicists, there are two kinds of matter in our universe: bosons and fermions. Bosons include, among many others, photons (particles of light), W bosons and Z bosons (which mediate the weak force), gluons (which mediate the strong force), and gravitons (which mediate gravity). Fermions include, among many others, electrons, neutrinos, and quarks.

Bosons are described by ordinary numbers and fermions are described by Grassmann numbers. Ordinary numbers have the property that a times b is equal to b times a, while Grassmann numbers have the property that a times b is equal to negative b times a. This implies that a Grassmann number times itself equals zero. a times a would have to be equal to negative a times a, so a times a must be zero ($0 = -0$).

This fact has a far-reaching consequence. I will return to this point soon. Meanwhile, let me explain what Pauli's exclusion principle is. Pauli came up with his exclusion principle to explain the periodic table. Although there are a lot of states that electrons can occupy in an atom, he concluded that no more than one electron in an atom can occupy the same state; there can be at most one electron per state. To ease your understanding, I will make the analogy that the state is like a seat, and the electron is like a person. At most one person can sit in a given seat. Two or more persons cannot sit in the same seat.

This exclusion principle can be easily understood from the mathematics as follows. An electron is a fermion, so it must be described by a Grassmann number. Let's say that a Grassmann number "$a(n)$" describes an electron in the nth state. Then, an electron in the nth state and another electron in the mth state should be described by the number $a(n)$ times $a(m)$. At this level, it would be hard to explain why we should multiply the numbers this way, so let's take it for granted. Now, let's consider the case that there are two electrons in the nth state. In this case, they should be described by the number $a(n)$ times $a(n)$. Lo and behold! We know that $a(n)$ times $a(n)$ is zero, since $a(n)$ is a Grassmann number. So, we see that two electrons cannot occupy the same state. The value zero means that there is no such possibility. Similarly, three or more electrons can be described by the number "$a(n) \times a(n) \cdots \times a(n)$". However, they are zero, as $a(n) \times a(n) = 0$, and 0 times $a(n)$s are always zero. So, we conclude that at most one electron can be in a given state.

Now, let's talk about the nitrogen nucleus and the discovery of neutron. It is easy to see that products of pairs of Grassmann numbers behave like ordinary numbers. For example, let's consider the four Grassmann numbers a, b, c, d. We have the following:

$$(ab)(cd) = a(bc)d = a(-cb)d = -acbd = (-ac)(bd) = (ca)(-db) = c(-ad)b = c(da)b = (cd)(ab) \tag{1}$$

So we conclude $(ab)(cd) = (cd)(ab)$. This completes the demonstration. One can extend this argument to show that a product of an even number of Grassmann numbers behaves like an ordinary number while a product of an odd number of Grassmann numbers behaves like a Grassmann number.

This implies that an even number of fermions behave like a boson, while an odd number of fermions behave like a fermion. Historically, a paradox following from this observation was resolved by the discovery of the neutron.

Before the neutron was discovered, physicists thought that a nucleus was made out of only protons and electrons. Therefore, physicists used to think that a nucleus of nitrogen was composed of 14 protons and 7 electrons. (They could find this out from the mass and the charge of a nitrogen nucleus.) However, this posed as a paradox. As both a proton and an electron are fermions, the implication was that a nitrogen nucleus should be a fermion, as there are 21 fermions in the nitrogen nucleus. (Remember that 21 is an odd number.) However, experiments showed that a nitrogen nucleus is a boson. This paradox was resolved with the discovery of the neutron. The nucleus of nitrogen is made out of 7 protons and 7 neutrons. So it is composed of 14 fermions, which correctly implies that a nitrogen nucleus is a boson.

Problem 1. Let a, b, c, d, and e be Grassman numbers. Which of the followings are equal to $abcde$? Which of the followings are equal to $(-abcde)$?

$$(a)\ eabcd \qquad (b)\ cadbe \qquad (c)\ cbade \qquad (d)\ edabc \qquad (e)\ aebcd$$

We see here that the five Grassman numbers are permuted. Those permutations equal to $abcde$ are called "even permutations, while those ones equal to $(-abcde)$ are called "odd permutations. For example, if you solve this problem correctly, you will see that $eabcd$ is an even permutation of $abcde$ while $cadbe$ is an odd permutation of $abcde$. We will encounter the concept of even permutation and odd permutation again in our later article "Levi-Civita symbol.

Summary

- Pauli's exclusion principle says that no more than one electron in an atom can occupy the same state.
- An electron is a fermion, so it must be described by a Grassmann number. As two electrons in the nth state are described by $a(n) \times a(n) = 0$, two electrons or more cannot occupy the same state.
- An even number of fermions behave like a boson, while an odd number of fermions behave like a fermion.