Dirac's bra-ket notation

Dirac's bra-ket notation is frequently used to denote vectors in quantum mechanics. As it is not difficult, you will benefit by learning it; it will facilitate reading quantum mechanics books. This article was written for students who have some understandings of linear algebra.

There are two kinds of vectors in bra-ket notation: bra vectors and ket vectors. A bra vector $\langle v|$ denotes a row vector, such as $\begin{pmatrix} v_x & v_y & v_z \end{pmatrix}$. A ket vector $|v\rangle$ denotes a column vector such as $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$. Therefore a bra vector and a ket vector are dual to each other; (i.e. they can be paired together to produce a scalar) I will show you shortly how one can denote a dot product between vector $\vec{A}$ and vector $\vec{B}$ in bra-ket notation. Let's first denote this product in matrix notation. It would be

$$\begin{pmatrix} A_x & A_y & A_z \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = A_x B_x + A_y B_y + A_z B_z \tag{1}$$

You can easily see that the row vector is on the left side and the column vector is on the right side. Therefore, you have to write a bra vector on the left, and a ket vector on the right. Then it would be $\langle A|B\rangle$. Of course, you may wonder why it has not been written as $\langle A||B\rangle$ but, it's just a convention that you write only one bar in the middle. Now, I will show you how you can write the completeness relation in terms of bra-ket notation. A completeness relation in three dimensions in matrix notation is as follows.

$$\begin{aligned} I &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

You may wonder how this sum could be an identity matrix.

You can easily see it, by multiplying by a row vector $\begin{pmatrix} v_x & v_y & v_z \end{pmatrix}$ on the left-hand side of the sum. Then you get

$$v_x \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} + v_y \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + v_z \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} v_x & v_y & v_z \end{pmatrix} \tag{2}$$

Similarly, you can multiply by a column vector on the right-hand side of sum, and you get the original column vector. Therefore, it is easy to see that this sum is an identity matrix. Here, you can see that the column vectors are on the left side of the row vectors. Therefore, ket vectors precede bra vectors in the completeness relation. So, you can write this as

$$I = \sum_{n=1}^{3} |e_n\rangle\langle e_n| \tag{3}$$

in bra-ket notation, where $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ denotes $\langle e_1|$, $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$ denotes $\langle e_2|$, and $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ denotes $\langle e_3|$.

Physicists frequently use the completeness relation in quantum mechanics. When there are infinite bases, and the index used to label bases is continuous (in above case, this index was n), the sum above is replaced by an integration. The most common examples are the position matrix x and the momentum matrix p. The completeness relation in these cases can be written as follows.

$$I = \int dx|x\rangle\langle x| = \int dp|p\rangle\langle p| \tag{4}$$

Here, $|x\rangle$ denotes the suitably normalized eigenvector with eigenvalue x. Similarly, $|p\rangle$ denotes the suitably normalized eigenvector with eigenvalue p. Here, I used the word "suitably" to reflect the fact that the eigenvectors with continuous eigenvalues must be normalized differently than the ones with discrete eigenvalues.[1]

Problem 1. Using our earlier notations for $|e_n\rangle$ and $\langle e_n|$, convince yourself of the following:

$$\langle e_i|H|e_j\rangle = H_{ij} \tag{5}$$

where H is a matrix and H_{ij} its components.

Remember that the components of identity matrix is given by the Kronecker delta symbol. Thus, we have

$$\langle e_i|I|e_j\rangle = \delta_{ij} \tag{6}$$

which implies

$$\langle e_i|e_j\rangle = \delta_{ij} \tag{7}$$

We can interprete this also as the following well-known formulas:

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \tag{8}$$

$$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{x} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{y} = \hat{z} \cdot \hat{x} = \hat{x} \cdot \hat{z} = 0 \tag{9}$$

If the basis of a vector space satisfies (7), we say the basis is orthonormal. Orthnormal means that two different basis vectors are orthogonal, and the norm (i.e., the length) of each basis vector is 1.

Let's use braket notation further. We can write a ket vector as

$$|v\rangle = \sum_i v_i|e_i\rangle \tag{10}$$

where v_is are the coefficients. And a bra vector as

$$\langle u| = \sum_i u_i\langle e_i| = \sum_j u_j\langle e_j| \tag{11}$$

[1] In other words, while the norm of $|e_n\rangle$ can be normalized to be 1 for any n, the norm of $|x\rangle$ can never be so for any x. We will obtain the suitable norm of $|x\rangle$ in our later article "A short introduction to quantum mechanics VI: position basis and the Dirac delta function."

Notice that it doesn't matter whether we use i or j as the variable. (Those of you who read "Einstein summation convention" will recognize this as the change of dummy variable.) Then, their dot product is given by

$$\begin{aligned} \langle u|v\rangle &= \sum_i \sum_j u_j v_i \langle e_j|e_i\rangle && (12) \\ &= \sum_i \sum_j u_j v_i \delta_{ji} && (13) \\ &= \sum_i u_i v_i && (14) \end{aligned}$$

Thus, we get the familiar result that the dot product is the sum of the products of coefficients.

Problem 2. Show that if (7) and (10) are satisfied, we have

$$\langle e_i|v\rangle = v_i \quad (15)$$

In other words, the coordinates of a vector in this basis are just the dot products of the vector with the basis vectors. This is not new. For example, if $v = 3\hat{x} + 4\hat{y} + 5\hat{z}$, we have $5 = \hat{z} \cdot v$. This works because $\hat{x}, \hat{y}, \hat{z}$ form a set of orthonormal basis.

Similarly, if (11) is satisfied, we have

$$\langle u|e_i\rangle = u_i \quad (16)$$

Given this, we can derive (14) in another way by using the completeness relation.

$$\begin{aligned} \langle u|v\rangle &= \langle u|I|v\rangle && (17) \\ &= \sum_i \langle u|e_i\rangle\langle e_i|v\rangle = \sum_i u_i v_i && (18) \end{aligned}$$

Of course, this is all tautology, but it is necessary to get used to the bra-ket notation.

Problem 3. Consider three-dimensional vector space, which has a set of basis $|e_i'\rangle$, where $i = 1, 2, 3$. This set of basis is not necessarily equal to $|e_i\rangle$, the one we considered in this article. Show that we can express the completeness relation as follows

$$I = \sum_{n=1}^{3} |e_n'\rangle\langle e_n'| \quad (19)$$

if and only if our basis is an orthonormal basis. (Hint[2])

Problem 4. Consider a vector space of which the basis is given by $|1\rangle$, $|2\rangle$, $|3\rangle$. Let's say that a linear operator A satisfies the following:

$$A|1\rangle = -|1\rangle + |2\rangle + |3\rangle, \qquad A|2\rangle = 3|1\rangle - 2|3\rangle, \qquad A|3\rangle = 4|1\rangle - |3\rangle \quad (20)$$

Then, express A in terms of a matrix.

Summary

- $\langle v|$ is a bra vector, and $|v\rangle$ is a ket vector.
- An inner product between a bra vector $\langle u|$ and a ket vector $|v\rangle$ is denoted as $\langle u|v\rangle$.
- The completeness relation is given by $I = \sum_i |e_i\rangle\langle e_i| = \int dx|x\rangle\langle x| = \int dp|p\rangle\langle p|$.
- A set of orthonormal basis satisfies $\langle e_i|e_j\rangle = \delta_{ij}$.

[2] Recall that a linear map is completely determined by how it acts on its basis. So, if a linear map A satisfies $A|e_i'\rangle = |e_i'\rangle$ for all i, A is an identity matrix.