Why are there as many even natural numbers as natural numbers?

What are the natural numbers? 1, 2, 3, 4, 5, 6, and so on. What are even natural numbers? 2, 4, 6, 8, 10, 12, and so on. So, it seems that the number of natural numbers is bigger than the number of even natural numbers, as there are natural numbers that are not even natural numbers. Namely, odd natural numbers: 1, 3, 5, 7, 9, 11, 13, and so on. You may perhaps think that there are twice as many natural numbers as even natural numbers, as the number of odd natural numbers is equal to the number of even natural numbers, and a natural number is either an even number or an odd number.

However, surprisingly, there are as many even natural numbers as natural numbers. Let me explain why. Read our article "One-to-one correspondence." I explained there that the number of elements of two sets is equal if there is a one-to-one correspondence between these two sets. So, if we find a one-to-one correspondence between even natural numbers and natural numbers, our job is done. Finding such a one-to-one correspondence is easy. Here is the correspondence: (even natural number $\leftrightarrow$ natural number)

$$2 \leftrightarrow 1, \quad 4 \leftrightarrow 2, \quad 6 \leftrightarrow 3, \quad 8 \leftrightarrow 4, \quad 10 \leftrightarrow 5$$
$$12 \leftrightarrow 6, \quad 14 \leftrightarrow 7, \quad 16 \leftrightarrow 8, \quad 18 \leftrightarrow 9, \quad \cdots$$

Thus, we indeed proved that there are as many even natural numbers as natural numbers. Unsatisfied? Do you still think that there are more natural numbers than even natural numbers? Recall our explanation in "one-to-one correspondence" and see if there are any natural numbers that can't be paired with even natural numbers. Just give me any natural number, then I will pair it with an even natural number. Four billion five? If I multiply it by 2, the even natural number that I give back is I get eight billion ten. I can always multiply *any* natural number by 2.

This proof was due to the German mathematician Georg Cantor in the 19th century. Just as you may find the conclusion of this proof hard to believe, many mathematicians in Cantor's day opposed Cantor's ideas. However, after his death, nobody disputes his proofs or ideas anymore. Certainly, unintuitive things happen when the number of elements in a set is infinity. Cantor also proved that the number of rational numbers is equal to the number of natural numbers.[1] An infinite set that has the same number of elements as the natural number is called "countable." For example, rational numbers are countable.

As Cantor proved, the number of irrational numbers is bigger than the number of rational numbers, which are countable. Therefore, the irrational numbers are "uncountable." He also proved that the number of (real) numbers between 0 and 1 is equal to the number of (real) numbers.[2] This is surprising because most real numbers lie outside the range from 0 to 1.

If the number of real numbers is equal to the number of real numbers between 0 and 1, which is a very small range, a natural question to ask is "Is there a number bigger than the number of natural numbers but smaller than the number of real numbers?" Or more precisely, "Is there a subset of real number, whose number of elements is bigger than the number of natural numbers but smaller than the number of real numbers?" Cantor thought that the answer was no, but failed to prove this. In 1940, Kurt Gödel proved that the existence of such a set cannot be proven. In 1963, Paul Cohen proved that the non-existence of such a set cannot be proven.

I could have explained more in detail, but they are not terribly interesting to me, as they have no applications in physics. But, it is interesting to know what kind of problems a small minority of mathematicians work on. I would rather not, and am not capable.

Problem 1. Explain why there are as many even natural numbers as integers.

[1] Rational numbers are numbers that can be expressed as a/b where a and b are integers. For example, 34/45 is a rational number, while $\sqrt{2}$ are not rational numbers. Read our essay "$\sqrt{2}$ as an irrational number." Irrational numbers are numbers that are not rational.

[2] Real numbers are just ordinary numbers whose square is non-negative. For example, -1.5 is a real number, because its square 2.25 is not negative.