Cauchy-Riemann equations

Consider a function “f” from a complex number “$z = x + iy$” to a complex number “$u + iv$” as follows:

$$f(x + iy) = u + iv \tag{1}$$

where x, y, u, and v are real. Given this, suppose you want to differentiate this function. Recalling the definition of differentiation, we have:

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = f'(z) \tag{2}$$

If this limit exists, it implies that we get the same value for the above expression no matter in which direction h approaches 0. If h approaches 0 along the real axis, we have:

$$\lim_{x \to 0} \frac{f(z + x) - f(z)}{x} = \frac{\partial f}{\partial x} \tag{3}$$

If h approaches 0 along the imaginary axis, we have:

$$\lim_{y \to 0} \frac{f(z + iy) - f(z)}{iy} = \frac{1}{i}\frac{\partial f}{\partial y} \tag{4}$$

Since both are equal we have:

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \tag{5}$$

Plugging (1), the above becomes:

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \tag{6}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{7}$$

Therefore, we conclude:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{8}$$

These are called “Cauchy-Riemann equations.” We will see their power in the next article.

Let me recap. If a function $f(z)$ has a well-defined limit for $f'(a)$ we say the function $f(z)$ is “complex differentiable” (also called “analytic” or “holomorphic”) at a. Then, it satisfies Cauchy-Riemann equations at $z = a$.

Now, let's investigate what kind of functions are holomorphic. First, $f(z = x + iy)$ can be understood as $f(x, y)$. Considering the following,

$$x = \frac{z + z^*}{2}, \qquad y = \frac{z - z^*}{2i} \tag{9}$$

x and y are functions of z and z^*. As f is a function of x and y, f can be interpreted as a function of z and z^*, treated independently. Now, we introduce the following notation:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{10}$$

These are defined, so that the followings are satisfied:

$$\frac{\partial z}{\partial z} = \frac{\partial z^*}{\partial z^*} = 1, \qquad \frac{\partial z^*}{\partial z} = \frac{\partial z}{\partial z^*} = 0 \tag{11}$$

Problem 1. Check that a function $f(z, z^*) = u + iv$ is holomorphic (i.e. it satisfies Cauchy-Riemann equation), if following is satisfied:

$$\frac{\partial f}{\partial z^*} = 0 \tag{12}$$

Therefore, a holomorphic depends only on z and not on z^*. In addition to this condition, if the function is ordinarily differentiable (i.e. the usual one as opposed to the complex one), it's holomorphic. For example, $z^2 + zz^*$ is not holomorphic as it depends on z^*. (i.e. z^* appears in the function.) $z^2 + z^3$ is holomorphic since it doesn't depend on z^*. $1/z$ is holomorphic except for the point $z = 0$ where it's not ordinarily differentiable.

As an aside, a function is called anti-holomorphic, if it satisfies:

$$\frac{\partial f}{\partial z} = 0 \tag{13}$$

In such a case z doesn't appear in f and f is a function of z^* only. The concept of holomorphicity and anti-holomorphicity plays a crucial role in string theory.

Summary

- A complex valued function f is called "complexly differentiable," if the following limit
$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$
exists. i.e., no matter which direction you approach the limit $h \to 0$, the answer is unique.

- The complex differentiability implies Cauchy-Riemann equations.

- A complexly differentiable function is also called "holomorphic function." A holomorphic function can be expressed only in terms of z without z^*. i.e., it satisfies
$$\frac{\partial f}{\partial z^*} = 0$$