Cavalieri's principle and the volume of cones and sphere

What are the volumes of cylinder and cuboid? See Fig. 1 and 2. We can easily calculate their volume, because they have always the same cross section when cut horizontally. Thus, we can just multiply the base area by height to obtain the volume.

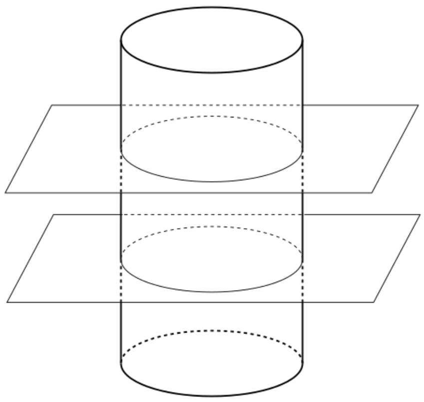

Figure 1: a cylinder has always the same cross section when cut horizontally

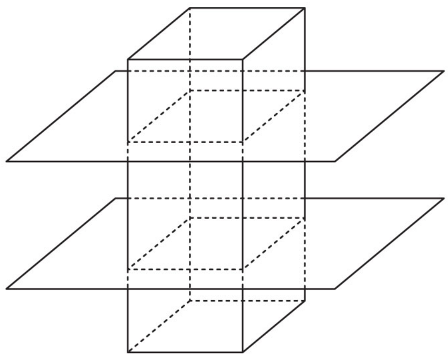

Figure 2: a cuboid has always the same cross section when cut horizontally

What are the volumes of cones? See Fig. 3 and 4. As they do not have the same cross sections, we cannot just multiply the base area by the height to get the volume. You can easily guess that the volume of cone is less than the base area multiplied by height, which is the volume of prism with the same base area and height, but at this point you cannot be sure about the exact volume. You may wonder whether you can multiply the base area by height and divide by 2 to obtain the volume, as you multiplied the base side by height and divide by 2 to obtain the area of triangle. Nevertheless, this is not the case as we will see. Anyhow, if you know calculus, it is very easy to derive the formula for the volume of cones or sphere. However, it is easy to lose insight on what is actually happening, if you just use calculus automatically without much thinking. Moreover, Archimedes had figured out the volume of cone and sphere about 2000 years before calculus was invented, so it is worth to learn how we can calculate them without using calculus.

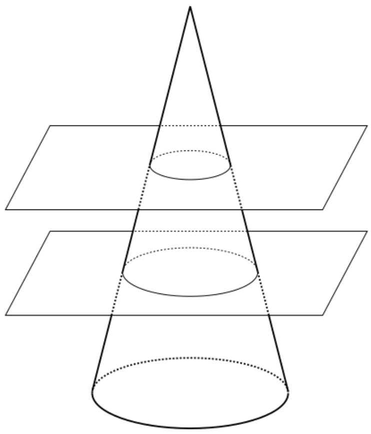

Figure 3: the cross section of circular cone when cut horizontally depends on the position of slice

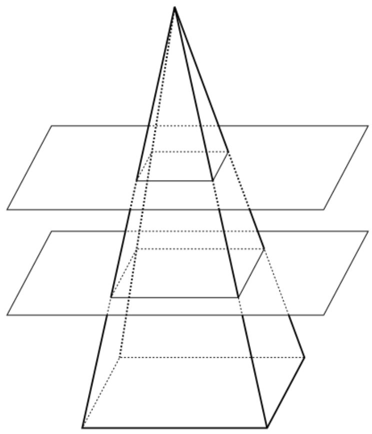

Figure 4: the cross section of rectangular cone when cut horizontally depends on the position of slice

1 The volume of square cone

Let's first begin with a very special type of cone. See Fig. 5 for a cube and six square cones. The blue point is located at the center of the cube. Why are there six square cones? The cube has six faces, each of which is the base of each square cone. These six square cones are exactly the same (congruent), so their volumes are also the same. As their total volume is the volume of cube, i.e., L^3, if each side is L, each cone has the volume of $L^3/6$.

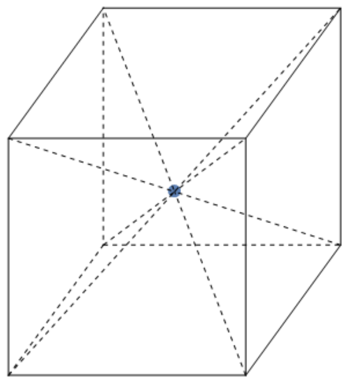

Figure 5: there are 6 cones in this cube

What are the base area and the height of each square cone? They are L^2 and $L/2$. Given this, let's consider a cone that has the same base, but different height, say L.

In other words, everything is the same as before except for the height which is doubled. If you remember how we calculated the area of an ellipse, you can easily figure out the volume of this cone.

Remember our earlier argument. A circle was fit in a square, and we fit this square into lattice, i.e., partitioning it into very small squares, and counted the number of small squares inside the circle, to calculate the area of circle. When a circle is stretched into an ellipse, along with the small squares, the small squares become stretched rectangles, and we can still count the number of the-square-turned-into-rectangles to calculate the area of the ellipse. This number is the same as the number of small squares we counted for the area of circle, because they were stretched together with the circle. However, what is different is now the area of each small rectangle. It changed by the ratio by which the circle is stretched. So, if we multiply the area of circle by the ratio by which it is stretched, we get the area of ellipse.

In our case now, i.e., 3-dimensional situation, the original square cone is inside a cube, and we fit this cube into lattice, i.e., partitioning it into very small cubes. By counting the number of small cubes that fit into the original square cone, we can obtain its volume. It should be $L^2/6$. Now, if we stretch the original square cone by double vertically, the-small-cube-turned-into-small-cuboid has the double volume. As the number of small cuboids inside the new square cone doesn't change, as both the small cubes and the square cone have been stretched by the same ratio, the total volume is simply doubled. Thus, we get that the total volume is doubled, i.e.,

$$\frac{L^2}{6} \times \frac{L}{L/2} = \frac{L^2}{3} \tag{1}$$

Problem 1. What is the volume of the square cone with base, the square with side L, and the height h?

If you correctly calculate this problem, you will get

$$\frac{1}{3} L^2 h \tag{2}$$

2 Cavalieri's principle

In elementary school, we learned that the area of triangle is given by its base multiplied by its height divided by 2. So, one may ask, why doesn't the area of triangle depend on its shape, as long as its base and height are the same? You surely learned the reason, but let's look at this from a slightly different perspective. See Fig. 6 for two triangles with the equal base and the equal height. We divided each rectangle into very small stripes with equal width. The area of each rectangle is the sum of the area of each stripe.

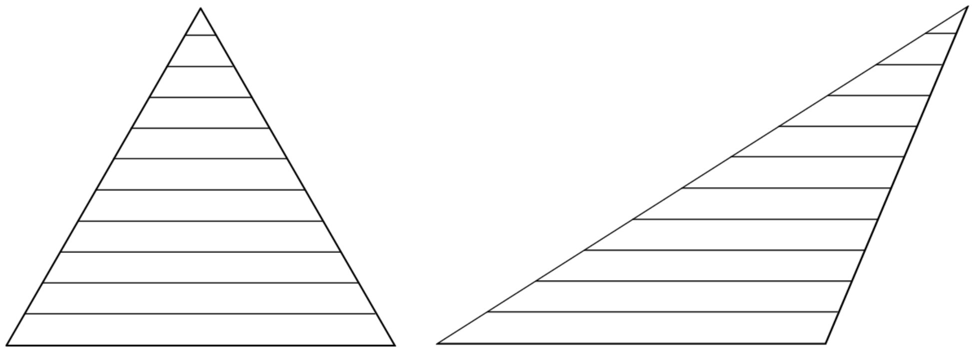

Figure 6: Two triangles with equal base and equal height. They are divided into stripes.

Now, let's compare the length of stripe. If you cut parallel the two triangles, the length of each stripe is always the same. See Fig. 7. The green lines have the same length, and the blue lines have the same length. As each corresponding stripe has the equal length, the sum of the area of each stripe is also the same. This is Cavalieri's principle.

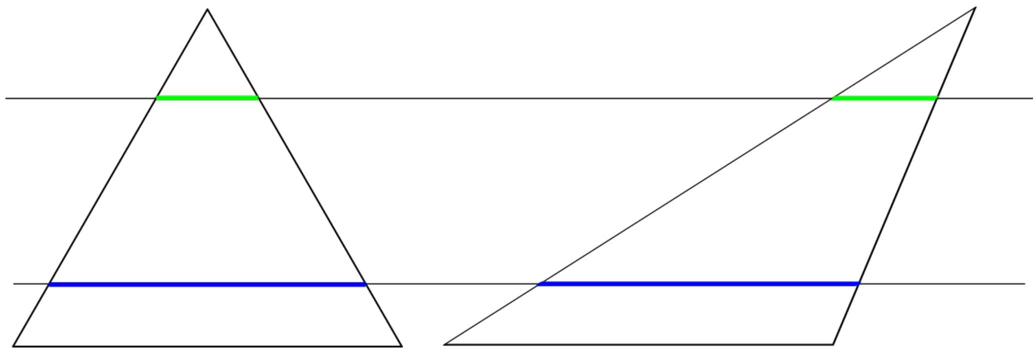

Figure 7: When cut horizontally, each slice has the same length

Let's see another example of the application of Cavalieri's principle. In Fig. 8, you see two cylinders, a usual cylinder, and a slanted cylinder. They have the same base area and the same height. If you cut them parallel, the cross section is excatly the same. See Fig. 9. As the volume is the sum of the area of each slice multiplied by its width, the two cylinders have the same volume.

This is easy to see if you see Fig. 10. The two stacks of coins have exactly the same volume, even though they are arranged in different shapes.

Cavalieri's principle applies even to the case in which the slices do not have the same shape. All that is required is that the slices have the same area. See Fig. 11 for a circular cone, and two rectangular cones. If they have the same base area, they have the same cross section area, when cut horizontally; the cross sections are smaller versions of the base, reduced by the same ratio. Thus, the three cones must have the same volume.

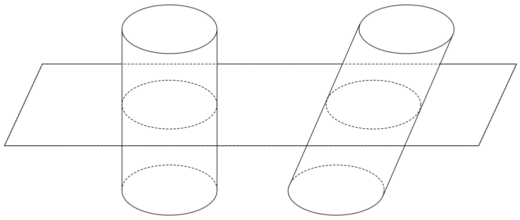

Figure 8: a normal cylinder and a slanted cylinder have the same volume, because they have the same cross section

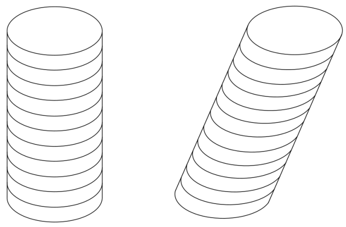

Figure 9: each of the two cylinders can be represented as a stack of disk. The area of each disk is the same

Figure 10: the volume of the normal cylinder on the left and the volume of “bent” cylinder are the same, because they have the same number of coins

Now, remember that we have obtained that the square cone with its base area L^2 and height h has the volume $L^2h/3$. Now, consider a cone, any type of cone, with base area A and height h. Then, its volume will be the same as the one of the square cone with its base area A and height h by Cavalieri’s principle. Let’s call the side of the base

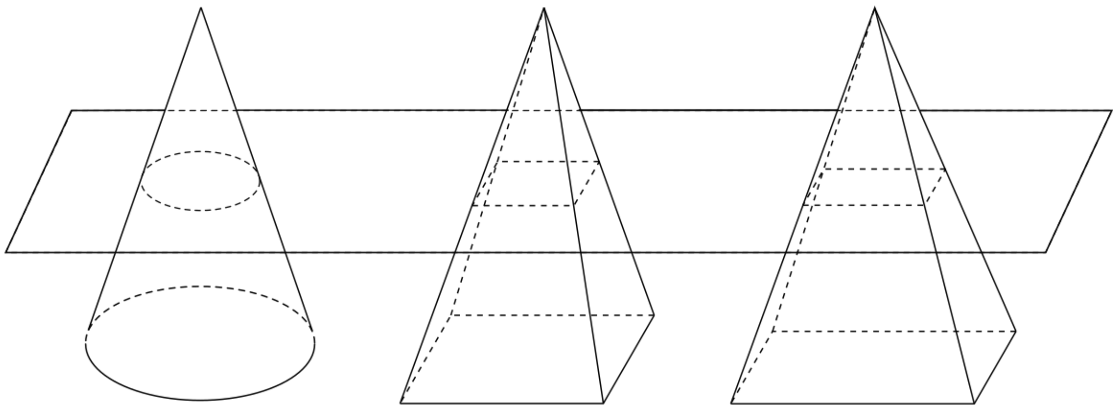

Figure 11: if the base of three cones have the equal area, the three cross sections when cut horizontally have always the same area. Thus, the volume of the three cones are the same.

of the square cone, l. Then, we have $A = l^2$. What was the volume of this square cone? It was $l^2h/3$. In other words, it is $Ah/3$. As the original cone has the same volume as the square cone, we conclude that the volume of the cone with base area A and height h is given by

$$V = \frac{1}{3}Ah \tag{3}$$

3 The volume of sphere

To calculate the volume of sphere, we first need to know the cross section area of sphere. See Fig. 12. If the sphere is cut y above from the center, the cross section has the area of $\pi(\sqrt{r^2 - y^2})^2 = \pi(r^2 - y^2)$ as you can check from the Pythagorean theorem.

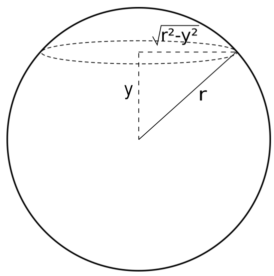

Figure 12: The radius of the cross section of sphere, cut y above the center is $\sqrt{r^2 - y^2}$

Now, we need to find an object that has the same cross section area as this one. See Fig. 13. On the left you see such an object. The region inside the cylinder, but outside the cone has the same cross section area. The cross section is given by the ring with the radius of outer circle r and the radius of the inner circle y. The ring is drawn in dotted line in the figure. Therefore, we see that the half sphere has the same volume as the object on the left. Its volume is given by the one of the cylinder subtracted by the one of the cone. Thus, the volume of half-sphere is given by

$$V(\text{half sphere}) = \pi r^2 \times r - \frac{1}{3}\pi r^2 \times r = \frac{2}{3}\pi r^3 \tag{4}$$

Thus, the volume of sphere is

$$V(\text{sphere}) = \frac{2}{3}\pi r^3 \times 2 = \frac{4}{3}\pi r^3 \tag{5}$$

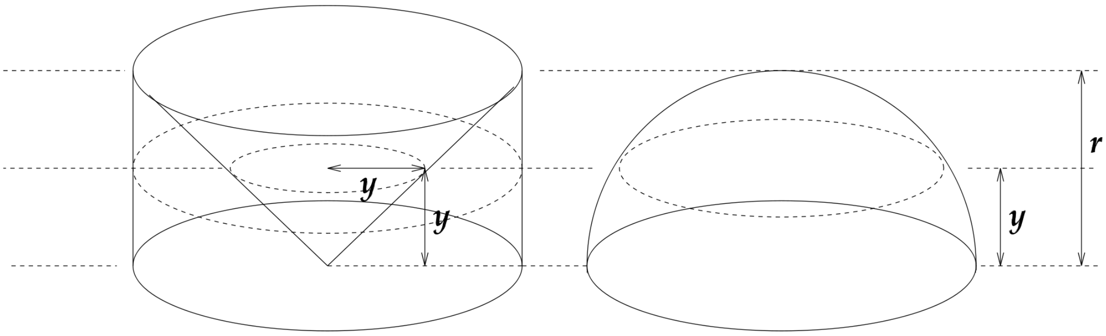

Figure 13: the volume of half-sphere is equal to the volume of the cylinder deducted by the volume of the cone

I only learned this method of calculating the volume of sphere long after I had learned how to calculate one using calculus. If I had never been interested in history of math, I would have never found it out.

Summary

- A cone with the base area A and the height h has the volume

$$V = \frac{1}{3}Ah$$

References

[1] `https://commons.wikimedia.org/wiki/File:Cavalieri%27s_Principle_in_Coins.JPG`