Center of mass

Suppose object 1 with mass 1 kg is located at the position (2 m, 0, 0) and object 2 with mass 1 kg is located at the position (6 m, 0, 0) and object 3 with mass 1 kg is located at the position (8 m, 0, 0), and object 4 with mass 1 kg is located at position at (12 m, 0, 0). Then what would be the "average" position, if such a thing exists? We can calculate it as follows:

$$(\frac{2\text{ m} + 6\text{ m} + 8\text{ m} + 12\text{ m}}{4}, 0, 0) = (7\text{ m}, 0, 0) \tag{1}$$

Now, suppose object 3 is located at position (6 m, 0, 0) instead of (8 m, 0, 0), and object 4 is also at position (6 m, 0, 0). What is the new average position? We have:

$$(\frac{2\text{ m} + 6\text{ m} + 6\text{ m} + 6\text{ m}}{4}, 0, 0) = (5\text{ m}, 0, 0) \tag{2}$$

Given this, notice that our new situation is equivalent to the case in which 1 kg is located at (2 m, 0, 0) and 3 kg is located at (6 m, 0, 0). Could we express the above formula slightly differently, using this? Certainly we can write:

$$(\frac{1\text{ kg} \times 2\text{ m} + 3\text{ kg} \times 6\text{ m}}{1\text{ kg} + 3\text{ kg}}, 0, 0) \tag{3}$$

This suggests us to define center of mass as follows. If an object i with mass m_i is located at $(x_i, 0, 0)$. The center of mass is:

$$\vec{r}_{cm} = (\frac{\sum_i m_i x_i}{\sum_i m_i}, 0, 0) \tag{4}$$

This is weighted average of the position of the objects. Convince yourself that this corresponds to (3) in our case. Notice also that this expression has the same structure as any weighted average. For example, if m_i students got the grade x_i for their midterms, the weighted average of the midterm grade would be exactly given by the above formula.

More generally, if the y and z coordinate is not zero, the center of mass is given by:

$$\vec{r}_{cm} = (\frac{\sum_i m_i x_i}{\sum_i m_i}, \frac{\sum_i m_i y_i}{\sum_i m_i}, \frac{\sum_i m_i z_i}{\sum_i m_i}) \tag{5}$$

If we re-express the object i's position as ($\vec{r}_i = (x_i, y_i, z_i)$, the above expression can be re-expressed as:

$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \tag{6}$$

Now, suppose the objects are moving. Then the center of mass is also moving. Let's calculate the velocity of the center of mass in terms of the velocity of objects. If an object i moves with velocity $\vec{v}_i = (v_{ix}, v_{iy}, v_{iz})$, its position is given by

$$\vec{r}_i = \vec{v}_i t + \vec{r}_{i0} \tag{7}$$

where $\vec{r}_{i0}$ is the initial position of the object i. Plugging this into (6), we get:

$$\vec{r}_{cm} = \left(\frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} \right) t + \frac{\sum_i m_i \vec{r}_{i0}}{\sum_i m_i} \tag{8}$$

We therefore see that the first term in the right-hand side of the above equation allows us the interpretation that the velocity of the center of mass is given as follows:

$$\vec{v}_{cm} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} \tag{9}$$

Summary

- The center of mass is given by

  $$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

  where m_i is each object's mass and $\vec{r}_i$ is each object's position.

- The velocity of the center of mass is given by

  $$\vec{v}_{cm} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i}$$

  where $\vec{v}_i$ is each object's velocity.