Center of mass of a half-ball

In an earlier article, we have obtained the following formula for the center of mass:

$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \tag{1}$$

where $\vec{r}_i = (x_i, y_i, z_i)$ is the position of ith object which has mass m_i. In this article, we will explicitly calculate the center of mass of a homogeneous (i.e. the density is constant) half-ball using this formula. See Fig. 1. for a half-ball. We will consider a half-ball instead of a ball as the center of mass of a ball is obvious; the center of mass of ball must be namely, the center of a ball i.e. right in the center from the spherical symmetry.

However, we encounter a difficulty when we try to directly apply (1). A half-ball is composed of very many small parts. It is almost impossible to count each part and sum them up by using (1). Therefore, we need to change the summation to an integration. This change is needed whenever we need to find the center of mass of body composed of very many small parts, just as we needed an integration to find the volume of such a body. See Fig. 2. If we denote each m_i as dm, (1) becomes

$$\vec{r}_{cm} = \frac{\int \vec{r}\, dm}{\int dm} = \frac{\int \vec{r}\, \rho dV}{\int \rho dV} \tag{2}$$

where ρ is the mass density and dV is the volume element. In our case, ρs in the numerator and the denominator cancel out, because ρ is a constant. Notice that the denominator of the above formula is the total mass of the body, just like the denominator of (1) is the total mass. This shows that we have suitably converted (1) to (2). Now, let's return to the case of half-ball. We will place the origin of our coordinate at the center of the ball, then the whole

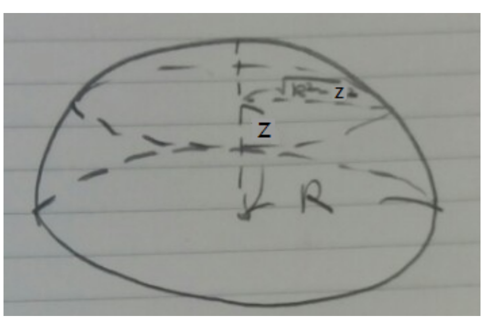

Figure 1: half-ball

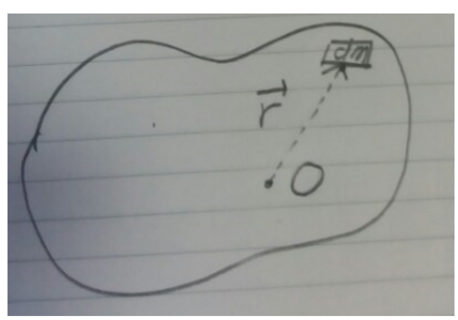

Figure 2: Finding the center of mass of a body composed of many small parts

half-ball is located at $z \geq 0$. From the symmetry, it is obvious that the x coordinate and the y coordinate of the center of the half-ball is 0. So, we only need to consider the z-coordinate.

As when we calculated the volume of a ball, we can regard the half-ball as a collection of very thin disks with thickness dz. Each thin disk has a volume $\pi(\sqrt{R^2 - z^2})^2$. Therefore, the z-coordiante of (2) becomes

$$z_{cm} = \frac{\int_0^R \pi z(R^2 - z^2)dz}{\int_0^R \pi(R^2 - z^2)dz} = \frac{3}{8}R \tag{3}$$

This is the answer. Notice that $\frac{3}{8}R$ is between 0 and R as expected. Actually, it is smaller than $\frac{1}{2}R$, because more mass is located in lower half part (i.e. $z < \frac{1}{2}R$) of the half-ball than in upper half part (i.e. $z > \frac{1}{2}R$) of the half-ball.

Problem 1. See Figure 4 of "Area inside a circle, surface area of a sphere and volume inside a sphere" where is the center of mass of a right circular cone with radius r and height h located?

Summary

- The center of mass is given by

$$\vec{r}_{cm} = \frac{\int \vec{r}\, dm}{\int dm} = \frac{\int \vec{r}\, \rho dV}{\int \rho dV}$$