Central force problem solution in terms of Lagrangian mechanics

In our previous article "The Lagrangian formulation of classical mechanics," we have explained how whole Newtonian mechanics can be re-written in terms of the Lagrangian picture. However, it seemed somewhat abstract since no example was given besides the most trivial one, namely, the equation of motion in 3d-Cartesian coordinate. In this article, we will solve the central force problem using Lagrangian mechanics, to give a concrete example and to show the vintage of using it.

Remember that Lagrangian is given by kinetic energy deducted by potential energy. In our earlier article "Kinetic energy in polar coordinate," we have seen that T, the kinetic energy in polar coordinate, is given by following:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{1}$$

Also, remember that the potential energy for the central force field (i.e. if the force is exerted toward or outward against the origin) should depend only on the distance from the origin. Therefore we can write V, the Potential energy, as follows:

$$V = V(r) \tag{2}$$

Then, the Lagrangian is:

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{3}$$

Then, for the equation of motion, we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \tag{4}$$

which yields:

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \tag{5}$$

We see that the first term in the right-hand side of the above equation is centrifugal force, and the second term is the central force.

We have another equation of motion. That is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \tag{6}$$

which yields:

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$
$$mr^2\dot{\theta} = const \tag{7}$$

which is the angular momentum conservation. It is fun that all these two equations can be easily calculated mathematically without resorting to complicated physical consideration.

Problem 1. Using Lagrangian, obtain the equation of motion for the pendulum considered in our earlier article "Simple pendulum." (i.e. $l\ddot{\theta} = -g\sin\theta$)

Summary

- In the central force problem with the potential energy that only depends on the distance to the origin,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$
leads to the equation of motion for the radial effective force which is a sum of centrifugal force and the central force due to the potential.

- And,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$
leads to the angular momentum conservation.