Centripetal force revisited with calculus

In our earlier article "Centripetal force," we obtained a formula for it using geometric ways. In this article, we will obtain the same formula again, using calculus.

Let's say an object moves around the origin with radius r, then we can write the location of the object (x, y) as follows:

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{1}$$

See Fig.1.

Since the object is moving at a constant speed, θ must be increasing at constant speed as well. Therefore, we can write as follows:

$$\theta = \theta_0 + \omega t \tag{2}$$

where θ_0 is the initial angle when $t = 0$. Now, let's calculate the velocity of this object. To this end, notice that the object's position can be written as follows:

$$x = r\cos(\theta_0 + \omega t), \qquad y = r\sin(\theta_0 + \omega t) \tag{3}$$

By differentiating, the velocity is:

$$v_x = \frac{dx}{dt} = -\omega r\sin(\theta_0 + \omega t), \qquad v_y = \frac{dy}{dt} = \omega r\cos(\theta_0 + \omega t) \tag{4}$$

The magnitude of the velocity is given by:

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2} = r\omega \tag{5}$$

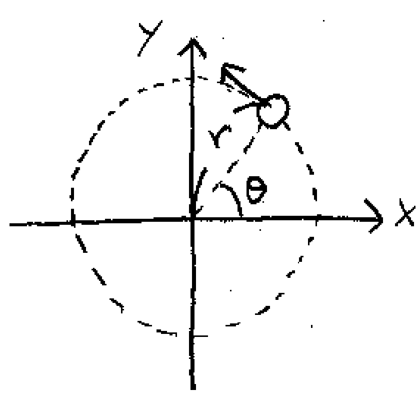

Figure 1: an object rotating

Now let's differentiate the velocity once more to obtain the acceleration. We have:

$$a_x = \frac{dv_x}{dt} = -\omega^2 r \cos(\theta_0 + \omega t), \qquad a_y = \frac{dv_y}{dx} = -\omega^2 r \sin(\theta_0 + \omega t) \tag{6}$$

The magnitude of the acceleration is given by:

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} = r\omega^2 \tag{7}$$

In terms of v, the above equation can be re-expressed using (5) as follows:

$$a = \frac{v^2}{r} \tag{8}$$

Moreover, by comparing (6) with (3), we see that the direction of the acceleration is opposite of the position (i.e. toward origin). Also, by comparing (4) with (3), we see that the direction of velocity is perpendicular to the position. Therefore, to move an object with mass m along such a circular orbit, with speed v, we need the following force:

$$F = ma = \frac{mv^2}{r} \tag{9}$$

Summary

- The centripetal force of an object with mass m moving with orbit radius r, with velocity v is given by

$$F = \frac{mv^2}{r}$$