The chain-rule

Suppose you want to calculate $(\sin 4x)'$. But, you can't, because we haven't taught you yet how to differentiate such a function. All we know is how to differentiate $\sin x$ or $4\sin x$ but not $\sin 4x$.

Therefore, in this article, we will teach you how to differentiate such functions called "composite functions." A composite function is a function obtained by successive applications of function. For example, if we have $y = g(x)$ and $z = f(y)$, we have $z = f(g(x))$. What we want to find in this article is $\dfrac{dz}{dx}$. In our case, we want to calculate $\dfrac{d\sin 4x}{dx}$, which says

$$z = \sin 4x = \sin y, \qquad y = 4x \tag{1}$$

In other words, z is a function of y which in turn is a function of x.

To differentiate a composite function, let's recall the definition of derivatives. We have:

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} \tag{2}$$

In other words, if x changes by a very small amount Δx, z changes by approximately $\dfrac{dz}{dx}\Delta x$. Of course this statement becomes exact when Δx is small.

Similarly, if x changes by a very small amount Δx. y changes by $\dfrac{dy}{dx}\Delta x$. Also, if y changes by a very small amount Δy, z changes by $\dfrac{dz}{dy}\Delta y$.

Now, we are ready to obtain $\dfrac{dz}{dx}$, which is our goal in this article. If x changes, y changes, which changes z in turn. If x changes by Δx, y changes by $\Delta y = \dfrac{dy}{dx}\Delta x$, which changes z by $\Delta z = \dfrac{dz}{dy}\Delta y = \dfrac{dz}{dy}\dfrac{dy}{dx}\Delta x$.

Therefore we conclude:

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \left(\frac{dz}{dy}\frac{dy}{dx}\Delta x\right)/\Delta x = \frac{dz}{dy}\frac{dy}{dx} \tag{3}$$

Loosely speaking, we can see as if dys in the numerator and denominator were canceled in the right hand side, even though your mathematician professors who too much care about rigor may not like putting it this way. This rule, which gives the derivatives of composite functions, is called "chain-rule."

Now, let's return to our example. Let's differentiate $z = \sin 4x$. If we write $y = 4x$, we have $z = \sin y$. Then,

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx} = \cos y \times 4 = 4\cos y = 4\cos 4x \tag{4}$$

One more example. Let's differentiate $z = \cos(x^2)$. If we write $y = x^2$, and take a similar step to our earlier example, you will get:

$$\frac{dz}{dx} = -2x\sin(x^2) \tag{5}$$

This is left as an exercise to the readers.

Problem 1. Find the derivatives of the following functions:

$$\frac{1}{10x+4}, \quad (3x+1)^{100}, \quad e^{\sin x}, \quad \sin(\cos x), \quad \frac{1}{\sin x}, \quad \sqrt{x^2+2x}$$

Problem 2. Show the following: (Hint[1])

$$(x^m(x+1)^n)' = x^{m-1}(x+1)^{n-1}((m+n)x+m)$$

Summary

- The chain-rule is given by
$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

[1] Use both the chain rule and Leibniz's rule.