Change of basis

You can specify a location in the two-dimensional plane by a two-dimensional vector starting at the origin and ending at the location. In such a situation, the two-dimensional vector can be expressed as "linear combination" of two (standard) "basis vectors" as follows:

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = v_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

Here, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the basis vectors.

But this is not the only way to specify the location - we could have used a different choice of basis vectors. For example,

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = v'_x \begin{pmatrix} 2 \\ 1 \end{pmatrix} + v'_y \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2}$$

or in other words,

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} v'_x \\ v'_y \end{pmatrix} \tag{3}$$

Here, we are using $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ as basis vectors. This is an example of "change of basis." We literally changed from one set of basis vectors (or "basis") to another. See Fig.1. In the first basis we have:

$$\vec{v} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

Whereas in the second basis we have:

$$\vec{v} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{5}$$

The second basis is just as good as the first one as far as expressing our vector is concerned. This is the logic that allows us to change basis. (However we should note that the dot product between two vectors $\vec{A}$ and $\vec{B}$ is not given by $A'_x B'_x + A'_y B'_y$ since the x'-axis and the y'-axis do not intersect in a right angle. One could choose a different basis that does preserves dot product, and in this case the x' and y' axes would intersect in a right angle.)

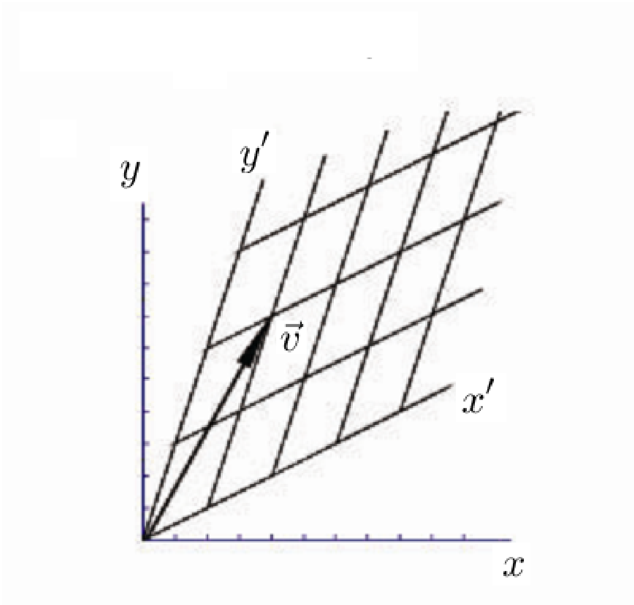

Figure 1: Change of basis

Change of basis plays a very important role in quantum mechanics. We will see that a given vector can be written in terms of different bases such as the "position basis," "momentum basis," "energy basis," etc.

Summary

- We can express the same vector in different sets of basis. This is called "change of basis."