Complex numbers

Let's try to solve the following equation:

$$x^2 = -9 \tag{1}$$

It is obvious that there is no solution to this equation. Certainly, no number squared is less than zero. Notice that the square of -3 is 9, not -9. Nevertheless, it is useful to introduce an imaginary number whose square is negative. Of course, such an imaginary number is not "real" in the usual sense. This is why it is called "imaginary." As a contrast, the ordinary number in usual sense is called "real" number.

This should remind you of your first encounter with negative numbers. There is no solution to $x + 4 = 3$ if we only consider non-negative solution for x. However, once we accept such x as existing, we can treat it as an actual number that satisfies the basic properties of arithmetic such as $x + y = y + x$, $xy = yx$, and $x(y + z) = xy + xz$. Same goes for imaginary number. Once we accept imaginary number as existing, we can treat it as an actual number that satisfies basic properties of arithmetic.

Furthermore, the benefit of imaginary number is far-reaching. For example, it is very useful in the mathematical analysis of electric circuit, and is essential for quantum mechanics, as we will see in later articles.

Having said this, let's define an imaginary number i as follows:

$$i^2 = -1 \tag{2}$$

Then we can obtain the solutions to the equation (1) in terms of imaginary number as follows:

$$x = 3i, -3i \tag{3}$$

This is obvious since

$$(3i)^2 = 3^2 i^2 = 9 \times (-1) = -9 \tag{4}$$

Similarly,

$$(-3i)^2 = (-3)^2 i^2 = 9 \times (-1) = -9 \tag{5}$$

$-3i$.

If we use the concept of the imaginary number, solutions to quadratic equations always exist. For example:

$$x^2 + 2x + 5 = 0 \tag{6}$$

doesn't have any solution that is a real number. However, it has solutions if we don't restrict ourselves to the case of real solutions and use the concept of imaginary number as follows:

$$\begin{aligned} (x+1)^2 &= -4 \\ x+1 &= 2i, \quad -2i \\ x &= -1+2i, \quad -1-2i \end{aligned} \tag{7}$$

Actually, we can check that they are the right solutions by substitution as follows. For $x = -1 + 2i$, we have

$$\begin{aligned} (-1+2i)^2 + (-1+2i) + 5 &= (-1)^2 + 2(-1)(2i) + (2i)^2 + 2(-1+2i) + 5 \\ &= 1 - 4i - 4 - 2 + 4i + 5 \\ &= 0 \end{aligned} \tag{8}$$

and similarly for $x = -1 - 2i$.

Such a number as x, which is a sum of a real number and an imaginary number is called a "complex number." A complex number "z" can be expressed as follows:

$$z = x + iy \tag{9}$$

where x and y are real numbers. x is called the "real part" of z, and y the "imaginary part" of z. We express this as $\mathrm{Re}z = x$, $\mathrm{Im}z = y$. By the way, a real number is also a complex number, as it can expressed in the form of (9). In such cases, y is zero.

You may wonder whether imaginary numbers exists, but they really do. I explained in the introduction to this article that imaginary numbers are widely used in quantum mechanics. Actually, apart from quantum mechanics, they are extensively used in science and engineering.

Notice that mathematicians didn't recognize the existence of the negative number as well before the 15th century. However, as we explained in our essay "The power of negative numbers," the concept of negative numbers is very powerful in solving cubic equations. The case with imaginary numbers is similar. In the 16th century, a general method to solving cubic equations was obtained. A cubic equation is an equation that can be expressed as follows:

$$ax^3 + bx^2 + cx + d = 0 \tag{10}$$

where a is non-zero.

This equation can have one, two, or three real solutions. By real solutions, I mean solutions that are real numbers. However, in the case where there is only one real solution, it sometimes happens you wouldn't be able to obtain this real solution without using the concept of imaginary numbers. The final result is a real number, but the intermediary step requires the use of the imaginary number to obtain the solution! Therefore, mathematicians began to realize that imaginary numbers really do exist. In Problem 8, we will show you the actual example.

Finally, let me explain the reason why i is neither greater nor less than $-i$. To show that i is not greater than $-i$, let's assume otherwise, and check that this assumption leads to a contradiction.

$$i > -i \tag{11}$$

$$i + i > -i + i \tag{12}$$

$$2i > 0 \tag{13}$$

$$i > 0 \tag{14}$$

As (14) says that i is positive, we can multiply i on both sides without changing the direction of inequality, thus

$$i \times i \quad > \quad 0 \times i \tag{15}$$

$$-1 \quad > \quad 0 \tag{16}$$

which is a contradiction. So we conclude that $i > -i$ is not true.

Problem 1. Show similarly that $i < -i$ is also not true.

Then, you might think that the only possibilty left is $i = -i$, but it would imply that $2i = 0$, which is a non-sense; if we square the both sides, we get -4 on the left-hand side while we get 0 on the right-hand side. -4 is not equal to 0. The conclusion is that it is non-sense to talk about which imaginary numbers are greater. In other words, imaginary numbers do not have "greatness" as real numbers do.

Problem 2.

$$(1+i)(1-4i) =?, \qquad i^{10} =?, \qquad \mathrm{Re}(3i) =?, \qquad \mathrm{Im}(4) =? \tag{17}$$

Problem 3.

$$\frac{1}{1+i} + \frac{1}{1-i} =?, \qquad \frac{1}{i^5} =? \tag{18}$$

Problem 4. Prove the following:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} \tag{19}$$

Problem 5. We have seen that the absence of the square root of a negative number motivated us to introduce imaginary numbers. Then, do we need to introduce another type of numbers to calculate the root of an imaginary number? Show that we don't, by checking

$$\left(\frac{1+i}{\sqrt{2}}\right)^2 = i \tag{20}$$

Thus, the square root of i can be expressed as a complex number. It is equal to $(1+i)/\sqrt{2}$. There are actually two complex numbers that satisfy $x^2 = i$. Can you find the other one? (Hint[1])

Problem 6. (Hint[2])

$$\left(\frac{1}{1+i}\right)^{10} =? \tag{21}$$

Problem 7. What is the real part of following? How about the imaginary part? (Hint[3])

$$\frac{1+5i}{1+i} \tag{22}$$

Problem 8. In 1572, Bombelli published a solution to $x^3 = 15x + 4$. For one of the real solutions, he obtained

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}} \tag{23}$$

Show the following: (Hint[4])

$$\sqrt[3]{2+\sqrt{-121}} = 2+i, \qquad \sqrt[3]{2-\sqrt{-121}} = 2-i \tag{24}$$

Thus,

$$x = 2+i+2-i = 4 \tag{25}$$

Summary

- $i^2 = -1$.
- A complex number is a number that can be expressed by a sum of a real number and a imaginary number. A real number is also a complex number.
- If $z = x + iy$ where x and y are real numbers, $\mathrm{Re}z = x$, $\mathrm{Im}z = y$.

[1] If x satisfies $x^2 = i$, it also satisfies $(-x)^2 = i$.
[2] $(1+i)^{10} = ((1+i)^2)^5$
[3] Use the result of Problem 4.
[4] Show $(2+i)^3 = 2+\sqrt{-121}$ and $(2-i)^3 = 2-\sqrt{-121}$.