Complex vector space

So far, we only considered real vector spaces. A real vector space is a vector space in which all the coefficients are real numbers. For example, a vector in an n-dimensional real vector space can be represented by n real numbers.

However, our nature is not as simple as real vector space. It turns out that complex vector space is necessary for quantum mechanics. In complex vector space, the coefficients are complex numbers. In other words, a vector in an n-dimensional complex vector space can be represented by n complex numbers.

This is not the only difference between real vector space and complex vector space. As we will see, the inner product (also called the dot product or the scalar product) needs to be defined slightly differently. In this article, I will explain why.

To this end, first let's think about how we can express the magnitude (also called "norm") of a vector using the bra-ket notation. The norm of $|u\rangle$ can be represented as follows:

$$\sqrt{\langle u|u\rangle} \tag{1}$$

For example, if $|u\rangle = 3\hat{x} + 4\hat{y}$, we have $\langle u|u\rangle = 25$, and $\sqrt{\langle u|u\rangle} = 5$.

Given this, we can now focus on complex vector space. Consider the following vector:

$$|v\rangle = (0.6 + 0.8i)|w\rangle \tag{2}$$

For simplicity, we will assume that the norm of $|w\rangle$ is 1. In other words,

$$\langle w|w\rangle = 1 \tag{3}$$

Now, let's calculate the norm of the state vector $|v\rangle$ considered above. The norm would be the square root of $\langle v|v\rangle$. If we naively calculate this, we have

$$\langle v| = (0.6 + 0.8i)\langle w|, \qquad |v\rangle = (0.6 + 0.8i)|w\rangle \tag{4}$$

so we get

$$\langle v|v\rangle = (0.6 + 0.8i)^2\langle w|w\rangle = (0.6 + 0.8i)^2 \tag{5}$$

Therefore the norm we get is

$$\sqrt{\langle v|v\rangle} = 0.6 + 0.8i \tag{6}$$

This is troublesome as the norm should be a non-negative real number. (There is no such thing as complex number valued magnitude.) However, we can get out of this dilemma by defining the norm of $|v\rangle$ to be the magnitude of $(0.6 + 0.8i)$ times the magnitude of $|w\rangle$. The magnitude of $(0.6 + 0.8i)$ is $\sqrt{0.6^2 + 0.8^2} = 1$, so the norm of $|v\rangle$ is 1. Another way of writing this is:

$$1 = \sqrt{(0.6 - 0.8i)(0.6 + 0.8i)} = \sqrt{(0.6 + 0.8i)^*(0.6 + 0.8i)} \tag{7}$$

where * denotes complex conjugation. In other words, if we define $\langle v|$ as $(0.6 + 0.8i)^*\langle w| = (0.6 - 0.8i)\langle w|$, we get

$$\langle v|v\rangle = (0.6 - 0.8i)(0.6 + 0.8i)\langle w|w\rangle = 1, \quad \rightarrow \quad \sqrt{\langle v|v\rangle} = 1 \tag{8}$$

Therefore, we conclude that if $|v\rangle = a|w\rangle$, then $\langle v| = a^*\langle w|$, as long as we want to make the norm of $|\vec{v}\rangle$ real.

One corollary that follows from this is that $\langle a|b\rangle^* = \langle b|a\rangle$. Let's try to prove this rigorously. Let

$$|a\rangle = \sum_i a_i|i\rangle, \qquad |b\rangle = \sum_i b_i|i\rangle \tag{9}$$

where $|i\rangle$ is an orthonormal basis. Orthonormality means

$$\langle i|j\rangle = \delta_{ij} \tag{10}$$

Recall our earlier article "Dirac's bra-ket notation." Then, as (9) implies

$$\langle a| = \sum_i a^*\langle i|, \qquad \langle b| = \sum_i b^*\langle i| \tag{11}$$

we have

$$\langle a|b\rangle = \sum_i (a_i^*)b_i \tag{12}$$

and

$$\langle b|a\rangle = \sum_i (b_i^*)a_i = \sum_i [(a_i^*)b_i]^*. \tag{13}$$

Therefore we easily see that indeed $\langle a|b\rangle^* = \langle b|a\rangle$. One corollary that directly follows from this is that $\langle v|v\rangle$ is real ($(\langle v|v\rangle)^* = \langle v|v\rangle$).

Final comment. In a later article, we will see that there is a vector that describes the state of a particle. This vector is called "state vector," or "wave function," and the vector space where such vectors live is called "Hilbert space."' Actually, in quantum mechanics, we only consider the case in which $\langle v|v\rangle$ is not only real, but also non-negative. However, when you learn quantum field theory or string theory, you will encounter the vectors whose $\langle v|v\rangle$ is not non-negative. Nevertheless, you will learn that such vectors don't correspond to the state vector of actual particles, and are thus called "ghosts." Eliminating ghosts is a big issue in string theory, and this can be only done in certain spacetime dimensions. (26 for bosonic string and 10 for superstring.)

Problem 1. Let $|a\rangle = (2+3i)|b\rangle$. Obtain $\langle a|$ in terms of $\langle b|$.

Problem 2. Let's say

$$\langle u|u\rangle = \langle v|v\rangle = 1, \quad \langle u|v\rangle = 0 \tag{14}$$

Then, what is the norm of the following vector $|\psi\rangle$?

$$|\psi\rangle = \left(\frac{1-i}{\sqrt{6}}\right)|u\rangle + \left(\frac{2}{\sqrt{6}}\right)|v\rangle \tag{15}$$

Summary

- If $|v\rangle = a|w\rangle$, then $\langle v| = a^*\langle w|$.
- $\langle a|b\rangle^* = \langle b|a\rangle$. Thus, $\langle v|v\rangle$ is always real as $\langle v|v\rangle^* = \langle v|v\rangle$.