Conic sections in Cartesian coordinate

As advertised in our earlier article "Conic sections and Newton's law of gravity," in this article we will prove the equivalence of two different definitions of ellipse, namely, one as a "squeezed circle" and another through foci. To this end, see Fig. 1. Let's say that the coordinate of two foci are given by $(-f, 0)$ and $(f, 0)$. Then, the sum of the distances to each focus is constant for an ellipse. What is this sum for our ellipse? $(a, 0)$ is a point on the ellipse. The distance to the first focus F_1 is $a + f$. The distance to the second focus F_2 is $a - f$. Therefore, their sum is $2a$. Now, we can write the equation for the ellipse. If (x, y) is a point on ellipse the distance to the first focus is given by $\sqrt{(x+f)^2+y^2}$ and the distance to the second focus is given by $\sqrt{(x-f)^2+y^2}$. Therefore, we have:

$$\sqrt{(x+f)^2+y^2}+\sqrt{(x-f)^2+y^2}=2a \tag{1}$$

This is the equation for the ellipse. Now, let's simplify it.

$$\begin{aligned}
\sqrt{(x+f)^2+y^2} &= 2a-\sqrt{(x-f)^2+y^2} && (2)\\
(x+f)^2+y^2 &= 4a^2-4a\sqrt{(x-f)^2+y^2}+(x-f)^2+y^2 \\
4a\sqrt{(x-f)^2+y^2} &= 4a^2-4xf \\
a\sqrt{(x-f)^2+y^2} &= a^2-fx \\
a^2(x-f)^2+a^2y^2 &= a^4-2a^2fx+f^2x^2 \\
a^2x^2-2a^2fx+a^2f^2+a^2y^2 &= a^4-2a^2fx+f^2x^2 \\
(a^2-f^2)x^2+a^2y^2 &= a^4-a^2f^2 && (3)\\
(\frac{x}{a})^2+(\frac{y}{b})^2 &= 1 && (4)
\end{aligned}$$

where in the last step we defined $b=\sqrt{a^2-f^2}$. The above equation shows that ellipse is a

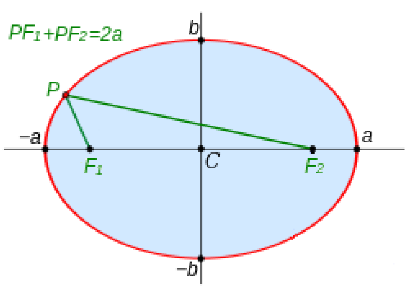

Figure 1: ellipse

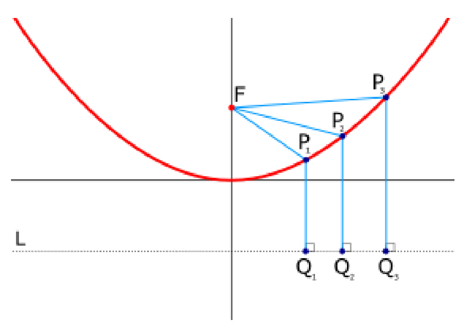

Figure 2: parabola

squeezed circle. A circle with radius a is given by the following equation:

$$(\frac{x}{a})^2 + (\frac{y}{a})^2 = 1 \tag{5}$$

If we squeeze this circle by the ratio a/b (i.e. rescale y by the ratio b/a), we have:

$$(\frac{x}{a})^2 + (\frac{y \times \frac{a}{b}}{a})^2 = 1 \tag{6}$$

which is precisely (4). This completes the proof. Also, this picture suggests how we could obtain the area of ellipse. Our circle has area πa^2. Since our ellipse is squeezed by ratio a/b, the area must be given by $\pi a^2/\frac{a}{b} = \pi ab$. This turns out to be correct, even though our derivation was by no means rigorous.

Now, having obtained a Cartesian coordinate expression for ellipse let's obtain one for parabola as well. See Fig.2. Let's say that the focus is at $(0, f)$ and directrix is given by $y = -f$. Then, we have:

$$\sqrt{x^2 + (y-f)^2} = y + f \tag{7}$$

Simplifying it, we obtain:

$$x^2 = 4fy \tag{8}$$

Finally, let's obtain a Cartesian coordinate expression for hyperbola. If the focus is at $(-f, 0)$ and $(f, 0)$. Then, we can write:

$$\sqrt{(x+f)^2 + y^2} + \sqrt{(x-f)^2 + y^2} = \pm 2a \tag{9}$$

Taking the similar step to the one that led to (3) one obtains:

$$(a^2 - f^2)x^2 + a^2y^2 = a^4 - a^2f^2 \tag{10}$$

$$(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1 \tag{11}$$

where this time we have $b = \sqrt{f^2 - a^2}$.

Therefore, we obtained Cartesian coordinate expressions for conic sections, albeit only in the cases in which the foci are on x-axis or y-axis. In general cases, the conic section can be always expressed in the following form.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ with } A, B, C \text{ not all zero} \tag{12}$$

This sounds reasonable, since (3), (8) and (10) are in this form. In our examples, Bs were always zero, but it's just because the foci are simultaneously on x-axis.

Even though we will not prove here, circle or ellipse corresponds to the case $B^2 - 4AC$ is negative, parabola corresponds to the case $B^2 - 4AC$ is zero, and hyperbola $B^2 - 4AC$ is positive. If this reminds you one of the formulas you memorized in middle school, you are correct. The number of solutions to a quadratic equation $ax^2 + bx + c = 0$ depends on the sign of $b^2 - 4ac$. Behind the criteria for which one of conic sections an equation corresponds to, the formula for the criteria for how many solutions are there to a quadratic equation is hidden there. Both of them are derived from the same principle.

Summary

- Conic sections can be analyzed in the Cartesian coordinate system.
- Conic sections are in the form of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ with } A, B, C \text{ not all zero}$$