Conic sections in polar coordinate

Having expressed conic sections in Cartesian coordinate, we will express them in polar coordinate. To this end, we will first consider an ellipse. It turns out that the expression is simplest when we choose one of the two focuses as the origin, and the other focus as lying at the coordinate $\theta = 180°$. See Fig.1. The semimajor axis is given by a and the semiminor axis is given by b. f' and f denote two focuses, which are offset from the center by ea. e is called "eccentricity." For circle e is 0, which means that two focuses are located at the same point, and for ellipse, it is less than 1 as the two focuses are always inside ellipse. We also know that $r' + r$ is constant. Let's try to express this quantity in terms of the semimajor axis or the semiminor axis. When the point concerned is at g, we have $r = a - ea$, $r' = a + ea$. Therefore, we obtain:

$$r + r' = 2a \tag{1}$$

When the point concerned is at h or i, we have:

$$r = r' = \sqrt{(ea)^2 + b^2} \tag{2}$$

which, in light of (1), implies

$$a = \sqrt{(ea)^2 + b^2} \tag{3}$$

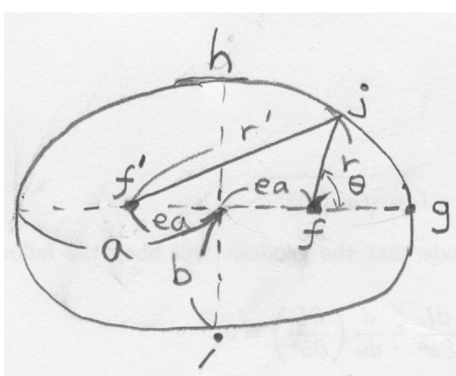
which, in turn, implies:

$$b = a\sqrt{1 - e^2} \tag{4}$$

Therefore, we obtained a relation between the semimajor axis a and the semiminor axis b in terms of the eccentricity e.

Now, let's obtain an equation for r' in terms of r and θ when the point concerned is j. We have:

$$r'^2 = (2ea + r\cos\theta)^2 + (r\sin\theta)^2 \tag{5}$$



$$r'^2 = r^2 + 4ea(ea + r\cos\theta) \tag{6}$$

Plugging $r' = 2a - r$, we get:

$$r = \frac{a(1-e^2)}{1 + e\cos\theta} \tag{7}$$

This is the expression for an ellipse in polar coordinate. It can be shown that, we have a hyperbola in the case $e > 1$.

Problem 1. Consider (7) and the case $e > 1$. What are the equations for asymptotes? What are the angles the two asymptotes make? Express using inverse trigonometric functions.

Problem 2. Why doesn't your answer in Problem 1 make sense when $-1 < e < 1$? (i.e. when it is ellipse or circle)

Summary

- Conic sections can be analyzed in polar coordinate.