Conic sections

Conic sections have been known since the ancient Greek era. However, it was not until the late 17th century that Sir Isaac Newton found their application to physics. In this article, we introduce conic sections.

The simplest and most well-known conic section is a circle. A circle consists of points that are located at equal distances from a center (see Fig. 1).

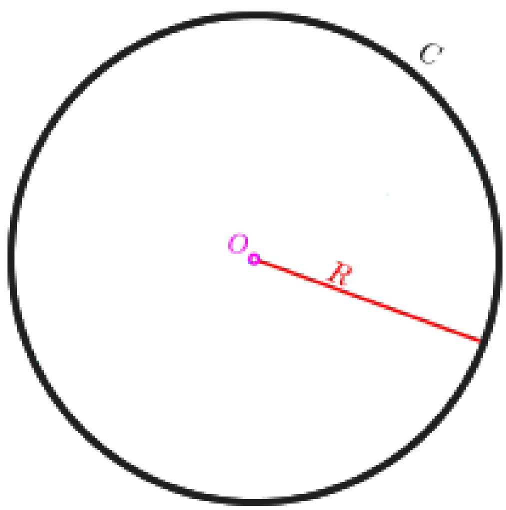

Figure 1: Representation of a circle. The point labeled as O corresponds to the center of the circle, R is the radius (the distance at which all points are located), and C is the curve comprised of such points. [1]

Our next example of a conic section is an ellipse (see Fig. 2). An ellipse has two foci, denoted in the figure as F_1 and F_2. An ellipse consists of all the points whose distances to each of the two foci add up to a fixed number. You see that this fixed number is $2a$ in our case. You can also draw an ellipse using this property. See Fig. 3.

Another way of understanding ellipses, even though it may be hard to draw one this way, is that an ellipse is a "squeezed" circle. Let me explain what this means. Let's say you have a circle on a Cartesian coordinates system (see Fig. 4). If you rescale the grid in the y direction (or the x, it doesn't matter) by a certain ratio, you have an ellipse (see Fig. 5). It is the beauty of mathematics that these two definitions of the ellipse are equivalent; the

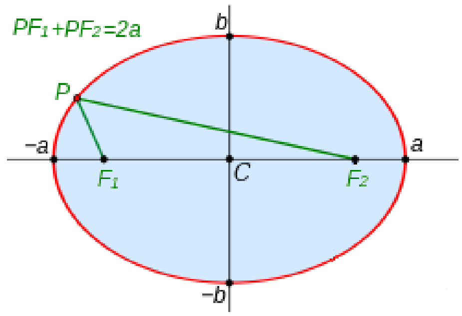

Figure 2: Same as Fig. 1 but with an ellipse. F_1 and F_2 are the two ellipse's foci, C its center, a and $-a$ are the horizontal coordinates of the intersection between the ellipse and the x axis, b and $-b$ are the corresponding intersections for the y axis, and P is a given point over the ellipse.

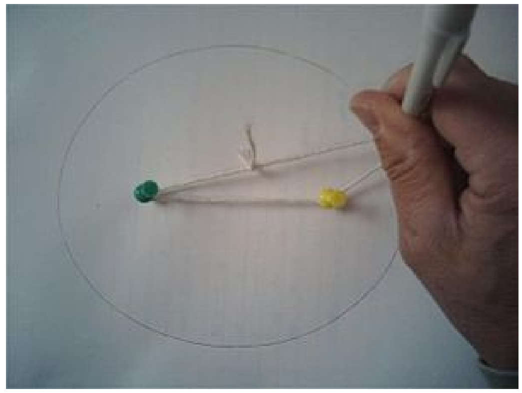

Figure 3: Drawing an ellipse. You can draw an ellipse by pinning the two ends of a piece of string to the foci, and drawing with a pen (or pencil), moving it around in such a way that the string is always tight. [2]

first one is through foci and the second one is through a squeezed grid. In our later article "Conic sections in Cartesian coordinates" we will prove this equivalence. Furthermore, and surprisingly, we will see yet another definition of an ellipse later in the article!

Notice also that a circle is a limiting case of an ellipse. If you take the two foci of an ellipse and move them closer and closer to each other, it will resemble a circle more and more. If the two foci coincide at a point, the ellipse becomes a circle, since the distances between a certain point on the ellipse and two foci will be the same, which means that the radius of the circle is simply given by the half of the sum of the distances to the two foci.

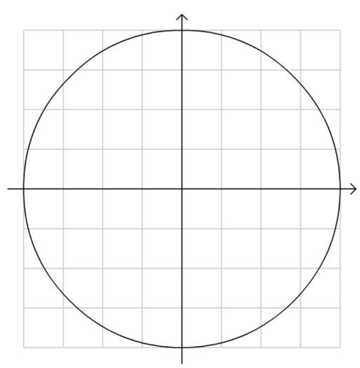

Figure 4: Schematics of a circle in a Cartesian coordinates system.

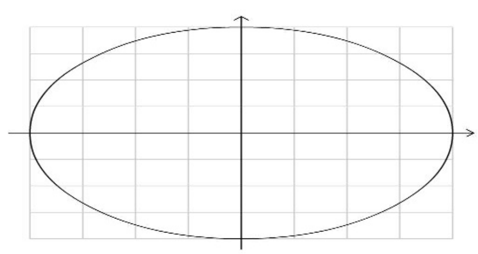

Figure 5: Rescaling of the x-component of Fig. 4, yielding a "squeezed" circle.

Our next example of a conic section is a parabola. A parabola has a focus and a line called a directrix. A parabola consists of all the points whose distances to the focus and to the directrix are the same (Fig. 6). It can be clearly seen in the figure that the distances FP_i and P_iQ_i with $i = 1, 2, \ldots$ are the same.

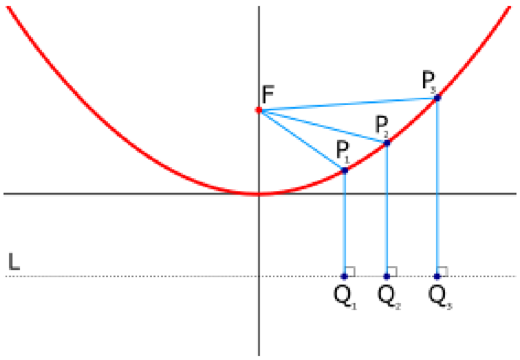

Figure 6: Representation of a parabola (red curved line). The focus is denoted as F and the directrix by L. P_i and Q_i, with $i = 1, 2, \ldots$, represent points on the parabola and the directrix, respectively.

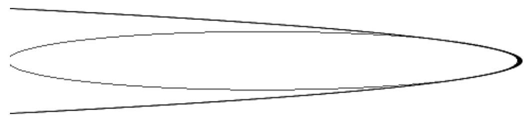

Figure 7: A parabola as a limiting case of an ellipse.

Even though it may not seem obvious, a parabola is actually a limiting case of an ellipse. If you fix one of the foci of an ellipse at a certain location and take the other one farther and farther from the first focus, this ellipse will resemble a parabola more and more (see Fig. 7). This relation will be obvious once we talk about yet another definition of an ellipse and a parabola later in this article!

Our last example of a conic section is a hyperbola. Like an ellipse, a hyperbola has two foci, however, its definition is completely different. It consists of all the points where the difference of the distances to the two foci is always the same (see Fig. 8).

Therefore, a hyperbola naturally has two separated parts, as you can see in the figure. As the difference of the distances is $2a$, the distance between the left focus and a point in the left part is $2a$ less than the distance between the right focus and that point. For the right part of a hyperbola, the opposite is true. Notice also that a hyperbola has two lines called "asymptotes", which are denoted by dotted lines in the figure. As a point in a hyperbola moves away farther and farther from the focus, it moves closer and closer to the asymptotes.

Even though it may not be obvious, a parabola is a limiting case of a hyperbola. If these two asymptotes become more and more parallel to each other (i.e. squeezed), the hyperbola resembles a parabola more and more (See Fig. 9). As we already mentioned before, this relation will be obvious once we talk about other definitions of a hyperbola and of a parabola!

As advertised, it is now time to introduce another definition for a circle, an ellipse, a parabola and a hyperbola. They are sections of a conic (see Fig. 10). A conic is represented

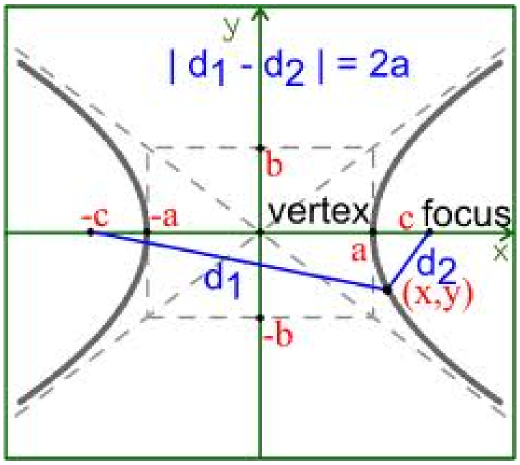

Figure 8: Representation of a hyperbola. The points at $x = c$ and $x = -c$ are the foci, $x = a$ and $x = -a$ correspond to the vertices, $x = b$ and $x = -b$ are the co-vertices, (x, y) are the Cartesian coordinates of an arbitrary point, and d_1 and d_2 are the distances from the foci to that point. [3]

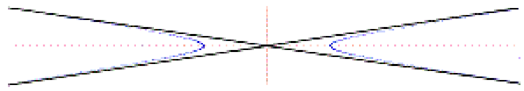

Figure 9: Graphical representation that the parabola is a limiting case of a hyperbola. The straight lines are asymptotes and the curved lines are a hyperbola.

in dark blue, and conic sections are in yellow. In the first figure, we see that we obtain a parabola if we cut a plane parallel to the side of cone. In the second figure, we see that we obtain a circle if we cut a plane horizontally. In the third figure, we see that we obtain a hyperbola. Notice that a hyperbola has two separated parts, as explained before.

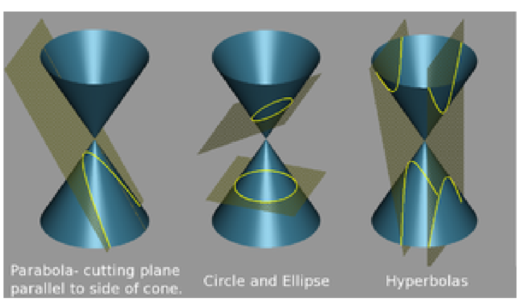

Figure 10: Graphical representation of a circle, an ellipse, a parabola and a hyperbola as the result of a section cut of a conic volume. [4]

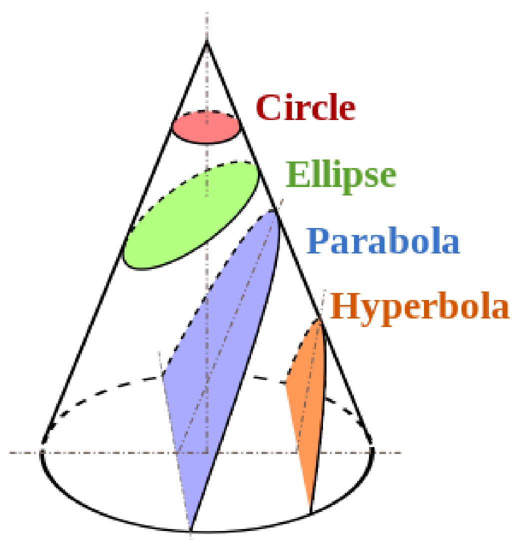

Figure 11: conic sections=sections of a conic

From these new definitions of conic sections, it is apparent that a parabola is a limiting case of an ellipse and of a hyperbola (see Fig. 11). If you cut the conic in a suitable direction you obtain an ellipse. If you cut it on a greater slant, actually parallel to the side of cone, you obtain a parabola. If you cut it on even more of a slant, you obtain a hyperbola (a hyperbola has actually two parts, but notice that in the figure we see only one, since only the lower part of the cone is drawn for convenience). Similarly, it is now clear that a circle is a limiting case of ellipse.

How these two definitions of conic sections (i.e. through foci and through a section of conics) are equivalent can be proved relatively easily, but students these days don't usually learn it. Nevertheless, as it is interesting, I included the proof for the case of an ellipse in our later article, "Ellipse, revisited."

There is another way to approach conic sections: algebraic method. Those of you who read our earlier articles "The center of mass of a triangle" or "The three altitudes of a triangle always meet at a point" will know what I mean by "algebraic method". We will approach conic sections this way in our later article "Conic sections in Cartesian coordinates" and "Conic sections in polar coordinates".

A final comment. In the early 17th century, Johannes Kepler discovered that the orbits of planets were all ellipses. In the late 17th century, Sir Isaac Newton explained why. We will talk more about this topic in later articles.

Summary

- A circle consists of points that are located at equal distances from a center.
- An ellipse consists of points whose distances to each of the two foci add up to a fixed number. It is also a "squeezed" circle.
- A parabola consists of points whose distances to the focus and to the directrix are the same.
- A hyperbola consists of points where the difference of the distances to the two foci is always the same.
- A circle, an ellipse, a parabola and a hyperbola are sections of a conic.

References

[1] `http://en.wikipedia.org/wiki/File:Circle-withsegments.svg`

[2] `http://en.wikipedia.org/wiki/File:Drawing_an_ellipse_via_two_tacks_a_loop_and_a_pen.jpg`

[3] This figure was reproduced with permission of Prof. James Jones.

[4] http://en.wikipedia.org/wiki/File:Conic_sections_with_plane.svg

[5] http://en.wikipedia.org/wiki/File:Conic_Sections.svg