What are continuous functions?

Having introduced the concept of limit in the last article, we can now introduce the concept of continuous function. To this end, we have to introduce what "continuous" means. A function $f(x)$ is continuous at a, if the following is satisfied:

$$\lim_{x \to a} f(x) = f(a) \tag{1}$$

A continuous function is a function continuous at every point of its domain. If a function is not a continuous function, it is a discontinuous function. Since it may be somewhat hard to grasp the meaning of continuous function and discontinuous function, let us give you two examples of discontinuous function. The floor function of x, denoted $[x]$ and attributed to Gauss, is defined by the greatest integer equal to or less than x. For example,

$$[-3] = -3, \quad [-2.99] = -3, \quad [-2.001] = -3, \quad [1] = 1, \quad [2.999] = 2$$

See Fig.1 for the graph of $y = [x]$. From the graph, we see that the graph is disconnected whenever x is an integer. These are the points where the floor function is discontinuous. Let's check that this is indeed so from the definition of continuity.

For example, let's check whether (1) is satisfied when $a = 2$. When x is slightly less than 2, we have:

$$[1.999] = 1, \quad [1.9999] = 1, \quad [1.99999999] = 1 \tag{2}$$

On the other hand, if x is slightly greater than 2, we have:

$$[2.001] = 2, \quad [2.0001] = 2, \quad [2.00000001] = 2 \tag{3}$$

Therefore, we see that the left limit (i.e. the limit obtained by approaching from slightly smaller values than a) and the right limit (i.e. the limit obtained by approaching from slightly bigger values than a) are different; the left limit is 1 while the right limit is 2. As we cannot assign a single value for $\lim_{x \to 2}[x]$, we cannot but conclude that the limit doesn't exist. Therefore, we can say that (1) is not satisfied for our case, as its left-hand side is not defined; we conclude the floor function is discontinuous when $x = 2$. Similarly, we can easily see that the floor function is discontinuous for any integer x.

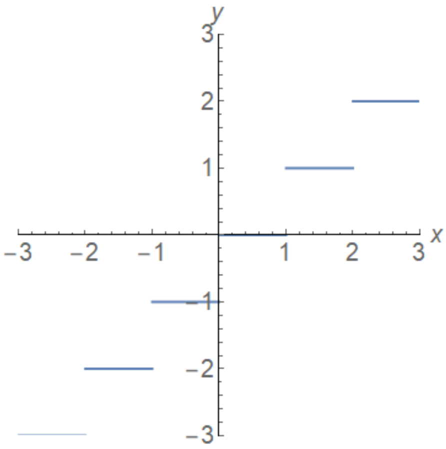

Figure 1: $y = [x]$

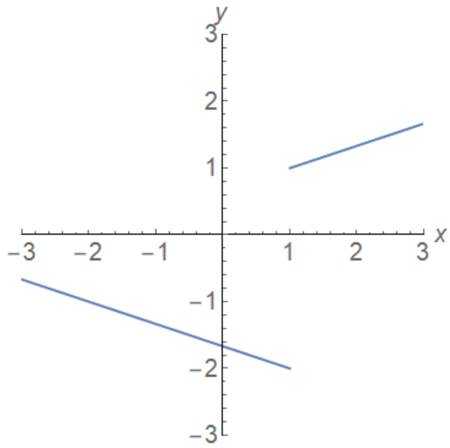

Figure 2: a function discontinuous at $x = 1$

Here is another example of a discontinuous function. See Fig. 2. You see that the graph is "broken" when $x = 1$. Thus, the left limit and the right limit when $x = 1$ are different, which means that (1) can't be satisfied for $a = 1$. Therefore, the function is discontinuous when $x = 1$.

The graphs of polynomial, sine and cosine functions are never "broken." Thus, they are good examples of a continuous function.

Problem 1. Is the absolute value function $|x|$ a continuous function? How about the sign function defined as follows?

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{4}$$

Summary

- A function $f(x)$ is continuous at a, if

$$\lim_{x \to a} f(x) = f(a)$$

- A continuous function is a function continuous at every point of its domain.

- Polynomials, sine functions and cosine functions are good examples of a continuous function.