Convexity, concavity and the point of inflection

We know what the physical interpretation of the first derivative. If $f'(x) > 0$ it means that $f(x)$ is increasing. If $f'(x) < 0$ it means that $f(x)$ is decreasing. In this article, we will find the physical interpretation of the second derivative.

Of course, if $f''(x) > 0$ it means $f'(x)$ is increasing. If $f''(x) < 0$ it means $f'(x)$ is decreasing. To find out the physical meaning, it helps to see some graphs of such examples.

See Fig.1. You see that $f'(x)$ is always increasing, which means that $f''(x) > 0$. Such a function is called "convex downward" or "concave upward."

See Fig.2. You see that $f'(x)$ is always decreasing, which means that $f''(x) < 0$. Such a function is called "concave downward" or "convex upward."

Sometimes, a function is concave upward or downward depending on its intervals. For example, see Fig.3. When $x < 1$ it is concave downward while it is concave upward when $x > 1$. In other words, the concavity (or equivalently, the convexity) changes when $x = 1$. Such a point (i.e. $(x = 1, f(x) = 0.5)$ is called "point of inflection."

Problem 1. Convince yourself that the following is satisfied for a convex downward function

$$f(\frac{a+b}{2}) < \frac{f(a)+f(b)}{2} \tag{1}$$

while the following is satisfied for a concave downward function.

$$f(\frac{a+b}{2}) > \frac{f(a)+f(b)}{2} \tag{2}$$

Hint[1]

Problem 2. Explain why $f''(a) = 0$ is necessarily satisfied if $(a, f(a))$ is the point of inflection.

Problem 3. Find all the points of inflection for the function $f(x) = x^4 - 6x^2 + 5x + 1$.

Problem 4. Explain why $(0, 0)$ is not the point of inflection for $f(x) = x^4$ even though $f''(0) = 0$ is satisfied.

Summary

- $f''(x) > 0$ means that the function $f(x)$ is "convex downward" or "concave upward."
- $f''(x) < 0$ means that the function $f(x)$ is "concave downward" or "convex upward."

[1] Draw a graph of an arbitrary convex downward function $f(x)$ and pick two arbitrary points on the graph $(a, f(a))$ and $(b, f(b))$. Then, the midpoint of these two points is given by $(\frac{a+b}{2}, \frac{f(a)+f(b)}{2})$. Mark this point on the graph and check whether this point is above or below $(\frac{a+b}{2}, f(\frac{a+b}{2}))$. Repeat this exercise for a concave downward function.

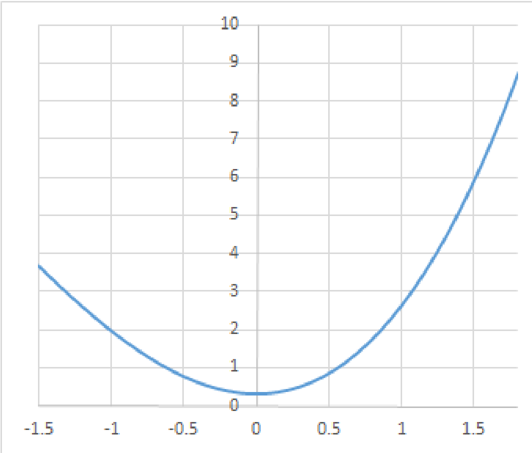

Figure 1: Convex downward

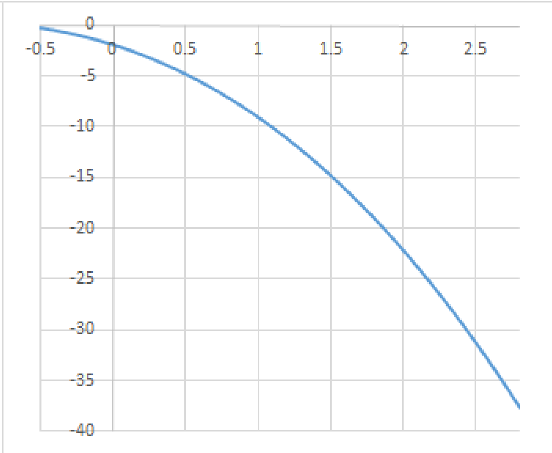

Figure 2: Concave downward

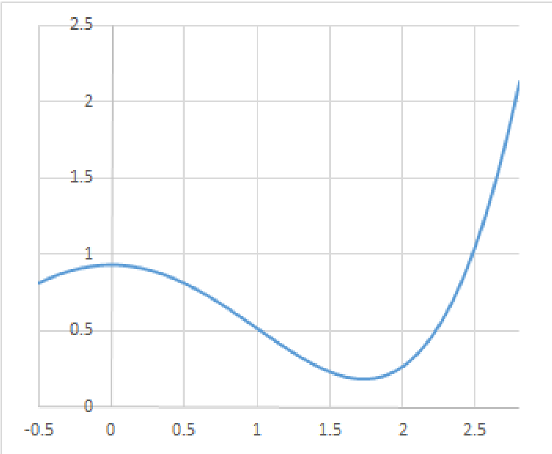

Figure 3: Point of inflection

- The point where convexity (or, equivalently, concavity) changes is called "point of inflection."