Coriolis force

As advertised in our earlier article "Centrifugal force," in this article, we introduce another type of inertial force present in rotating frame.

If a frame is rotating in anti-clockwise direction, from the point of view of an observer in the rotating frame, an object receives inertial force called "Coriolis force" upon moving, in the right direction with respect to the moving direction. Similarly, if a frame is rotating in clockwise direction, an observer sees the object receiving "Coriolis force" in the left direction with respect to the moving direction.

In this article, we will show this only in the case of anti-clockwise rotation, because the other case can be demonstrated similarly. In particular, we will consider two subcases: when the object is moving in the angular direction and when the object is moving toward the center of axis (i.e. opposite to radial direction).

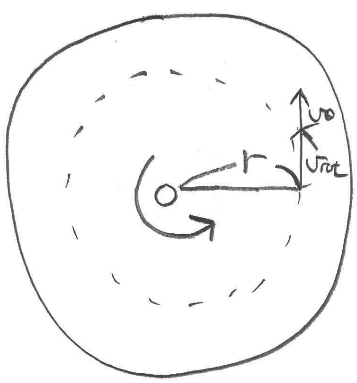

Fig. 1

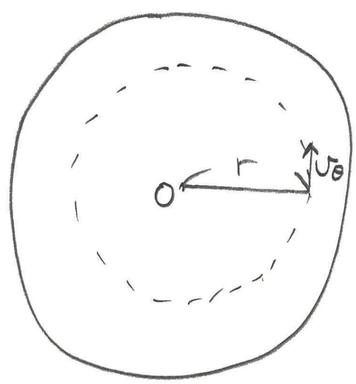

Fig. 2

For the first subcase, see Fig. 1, and Fig. 2. Fig. 1. shows the point of view of the inertial frame, and Fig. 2. shows the point of view of the rotating frame. You are at O, the center of the rotating disk, and you additionally rotate an object with a rope. The object is r meters away from you. The dotted lines denote the trajectory of the object. The velocity it would have had if the object is not moving with respect to the disk is v_{rot}. In other words, if the angular velocity of the rotating disk is ω, we have $v_{\text{rot}} = r\omega$. However, the object is moving with the additional velocity v_θ. In other words, from the point of view of the inertial frame, the object is rotating with the speed $v_{\text{rot}} + v_\theta$. Then, what is the tension of the rope? Notice that somebody in the inertial frame, and somebody in the rotating frame have to agree on the tension, even though they do disagree on the velocity of the object. If the tension is measured by something like a spring or scale, both observers must agree on the value the spring or the scale shows, as a number will be seen on a screen, which cannot be read differently.

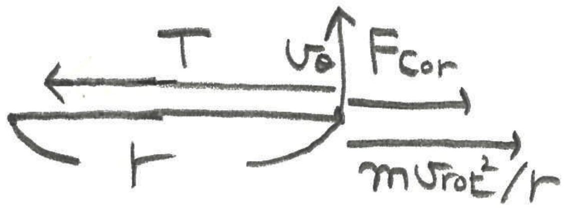

Fig.3

The tension is easiest to calculate in the inertial frame, because we do not need to consider the inertial force such as the centrifugal force or the Coriolis force. As the object is moving with the velocity $v_{\text{rot}} + v_\theta$, we have

$$T = \frac{m(v_{\text{rot}+v_\theta})^2}{r} \tag{1}$$

How can we interpret this in the rotating frame? The object is moving with acceleration of v_0^2/r toward the center. Therefore, the total force acting on the object is given by

$$F_{\text{total}} = mv_\theta^2/r \tag{2}$$

This force is provided by the tension and the inertial forces. Recall that the centrifugal force, a type of the inertial force, is given by mv_{rot}^2/r directing outward. So, we may naively think that the total force acting on the object is given by

$$F_{\text{total}} = T - \frac{mv_{\text{rot}}}{r^2} \tag{3}$$

However, if you actually plug in (1), you will see that you will not get (2). The additional inertial force you need to make the above equation right is the Coriolis force. See Fig. 3. Actually the Coriolis force acts outward in this case, as you will see shortly. Thus, we can write

$$F_{\text{total}} = T - \frac{mv_{\text{rot}}}{r^2} - F_{\text{Cor}} \tag{4}$$

Problem 1. Show that the Coriolis force F_{Cor} is given by

$$F_{\text{Cor}} = 2m\omega v_\theta \tag{5}$$

That we put $-$ sign in front of F_{Cor} in (4), and obtained a positive value for F_{Cor} means that the Coriolis force is indeed acting outward in this case. Thus, we see that the Coriolis force is acting in the right direction with respect to the moving direction in this case.

Second subcase. Fig. 5 shows the point of view of the inertial frame, and Fig. 6 below shows the point of view of the rotating frame. Let's say that the observer shoots a bullet toward the origin O with v_0 in the rotating frame. See Fig. 6. In the inertial frame, this translates to shooting a bullet askew with velocity given by vector addition of v_A and v_0. See

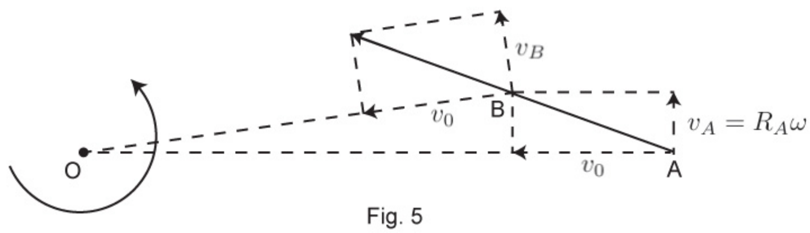

Fig. 5

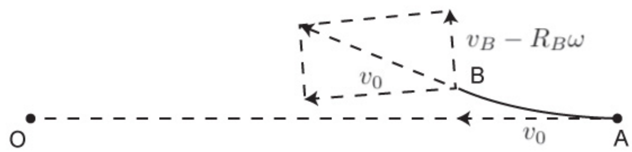

Fig. 6

Fig.5. When it reaches B, the radial velocity is v_0' which is quite close to v_0 if the distance between A and B is not big. So, we just denoted it as v_0 instead of v_0'. The angular velocity is v_B. We can calculate the v_B from angular momentum conservation. If the distance between O and A is R_A, and the distance between O and B is R_B, we have $mR_Av_A = mR_Bv_B$. Thus, we have $v_B > v_A$. How does this look in Fig. 6? At B the radial velocity is v_0' and the angular velocity is given by $v_B - R_B\omega$. (The angular velocity at A is $v_A - R_A\omega = 0$.) (**Problem 2.** Check that $v_B - R_B\omega > 0$.) As $v_B - R_B\omega > 0$, we see that the bullet turns rightward.

Problem 3. In case of Fig 5, show that the rightward acceleration due to the Coriolis force is approximately given by $2\omega v_0$. (Hint[1]) Therefore, the Coriolis force is given by $F_{\text{Cor}} = 2m\omega v_0$ in this case.

Summarizing, we have seen that the Coriolis force is given by

$$F_{\text{Cor}} = 2m\omega v_\theta \hat{r} \tag{6}$$

when moving with the velocity $v_\theta\hat{\theta}$, and by

$$F_{\text{Cor}} = 2m\omega v_0 \hat{\theta} \tag{7}$$

when moving with the velocity $-v_0\hat{r}$. In both cases, Coriolis force is acting in the right direction with respect to moving direction, and given by $2m\omega v$ where v is the moving speed.

When the object is moving in an arbitrary direction it is simply a linear combination of (6) and (7), and we get the result that the Coriolis force is acting in the right direction with respect to moving direction and is given by $2m\omega v$ where v is the moving speed. Of course,

[1] Use $R_A - R_B \approx v_0 t$. Here t is the time that the bullet took from A to B.

here v denotes only the velocity components parallel to the rotating plane. When you move along the axis of the rotation there is no reason to feel the Coriolis force.

From now on, let's talk about the rotation of the Earth as actual examples. Everybody knows that the Sun rises in the East and sets in the West. This means that the Earth rotates from the West to the East. This means that the Earth rotates anti-clockwise in Northern hemisphere and clockwise in Southern hemisphere. Therefore, an object moving is deflected rightwards in Northern hemisphere and deflected leftwards in Southern hemisphere. For example, the wind receives such a force, and particularly, the wind blowing toward the eye of typhoon is always deflected rather than going straight toward the eye of typhoon, which implies that typhoons look like spirals.

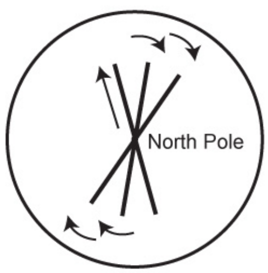

Fig. 8

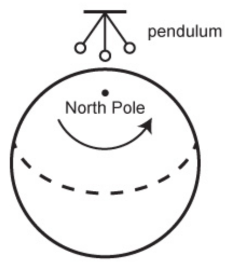

Fig. 7

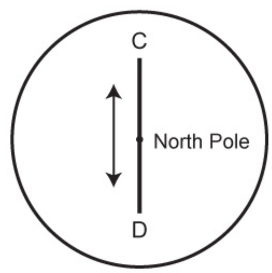

Fig. 9

See Fig. 7. A pendulum is situated right above the North Pole. In Fig.9 in which the point of view of inertial frame is depicted, if the pendulum oscillates between C and D nothing changes, as much as the trajectory in Fig.4 was straight. However, from someone moving with the Earth, the oscillation trajectory of the pendulum moves clockwise as the Earth is rotating anti-clockwise. See Fig.8. In addition, you see there that this is also consistent from the Coriolis force picture. The direction of the oscillation is denoted there and it is clearly deflected rightwards. In any case, it is easy to imagine that the trajectory of the oscillation comes back to the original position after 24 hours. (Exactly speaking, 23 hours and 56 minutes and 4 seconds, as explained in "History of Astronomy from the late 17th century to the early 20th century.")

Furthermore, if we situate Foucault's pendulum in northern hemisphere the trajectory of the oscillation will rotate clockwise, as it is evident from the Coriolis force picture.

Similarly, if we situate Foucault's pendulum in southern hemisphere the trajectory of the oscillation will rotate anticlockwise, as it is evident from the Coriolis force picture. Particularly, if the pendulum is situated at the South Pole, the trajectory of the oscillation will come back to the original position after 24 hours.

If Foucault's pendulum is situated in equator, it can rotate neither clockwise nor anti-clockwise. In other words, it will not rotate, or takes infinite time for the trajectory of the oscillation to come back to itself. From this point of view, Foucault's pendulum somewhere in the Northern hemisphere but not on the North Pole comes back to its original position between 24 hours and infinite time. In other words, it takes longer than 24 hours to come back to itself.

Problem 4. Which of the following is the correct answer for the hours a Foucault's pendulum at latitude θ takes to come back to its original position? (Multiple choice problem)
(a) $24\cos\theta$ (b) $24\sin\theta$ (c) $24\tan\theta$ (d) $24/\cos\theta$ (e) $24/\sin\theta$ (f) $24/\tan\theta$

We now come to our final point in this article. In our earlier essay, "Did Einstein really prove that Newton was wrong?" I promised you to explain why a ball vertically dropped falls slightly east contrary to our naive expectation based on the fact that the Earth rotates itself eastward. If you have already read this article so far you know why, but let me put it in words.

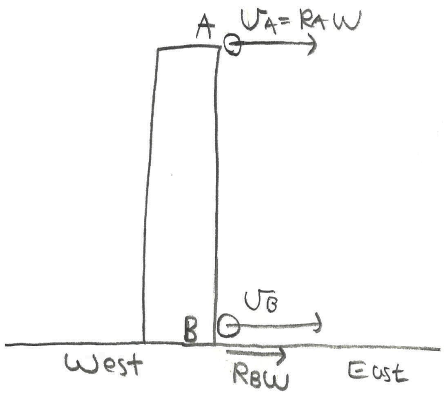

Fig. 10

See Fig. 10. You drop a ball from a rooftop of high building. As the ball is initially farther from the center of the earth than the ground is, it was already moving at a faster eastward speed than the ground. So, it is natural that the ball lands slightly east. (In the language of Fig. 5 and Fig. 6, $v_A = R_A\omega > R_B\omega$). However, this is not the whole story. As we have seen, the angular momentum conservation makes the eastward velocity of the ball bigger again by the time it reaches the ground. (In the language of Fig. 5 and Fig. 6, $v_B > v_A$.) The first effect corresponds to $a = \omega v_0$ and the second effect corresponds to $a = \omega v_0$. These two effects combined, we get $a = 2\omega v_0$, the acceleration due to the Coriolis force.

Problem 5. Suppose you are at the equator, and release a ball from the height h meter to the ground. Will it land on the west or on the east? How far west or how far east will it land? Assume that h is so small enough that the velocity of the ball is almost downward, which means you only need to consider the Coriolis force due to the virtical velocity component of the ball. You can regard the vertical velocity of the ball as $v = gt$ where t is the time elapsed after the drop.

Problem 6. Suppose now you shoot a ball vertically at the equator. Until the time it reaches the highest point, will it be tilted west or east?

Historical comments. French mathematician and scientist Coriolis published his idea on Coriolis force in 1835. He was also the first one to derive that the kinetic energy is given by

$\frac{1}{2}mv^2$. In 1851, French physicist Foucault demonstrated that the Earth rotates by showing that Foucault's pendulum rotates.

We will actually approach the Coriolis force more systematically in the next article.

Summary

- In a rotating reference frame, there are two inertial forces for a moving object: Coriolis force and centrifugal force. In this frame, if the object is not moving, there is no Coriolis force and it receives the centrifugal force only.
- Coriolis force acts on the direction perpendicular to the moving direction.
- The oscillating plane of Foucault's pendulum rotates due to Coriolis force. It doesn't rotate at all on the equator, and it rotates once every 24 hours on the North Pole or the South Pole.